Statistical mechanics of vortex lines

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Statistical mechanics of three-dimensional flows of an ideal incompressible fluid is considered. An ideal fluid differs from the usual Hamiltonian systems of statistical mechanics by possessing an infinite number of integrals of motion that are circulations of velocity over closed fluid contours. To reduce the problem to a standard one, the governing equations should be written in a Hamiltonian form in which all integrals of motion other than energy are eliminated. This is achieved by a generalization of the variational principle, which solves this problem in a two-dimensional case. The formulated variational principle can be interpreted as a variational principle for the dynamics of vortex lines. An invariant measure in the space of vortex lines is derived. For effectively two-dimensional flows this measure is reduced to the invariant measure obtained previously. As an example of application to effectively three-dimensional flows, the equation for averaged stream function for turbulent flow in pipes is derived. [S1063-651X(98)03803-3]

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I. INTRODUCTION

Turbulent motion of fluids remains a unique mechanical phenomenon that is still not understood from the first principles. A conceptual basis for turbulent theory has been laid down recently by the general theory of dynamical systems. From the perspective of the theory of dynamical systems, to develop a theory of turbulence means to find the probabilistic measure of the attractor of Navier-Stokes equations. The term ''probabilistic measure'' means the same as that in statistical mechanics: the probability of the event is the fraction of time during which this event is observed. The complexity of geometrical structure of attractors does not leave much hope for any possibility of theoretical prediction if the attractor is low dimensional. However, high dimensionality of fluid flows may result in developing an invariant measure that admits a simple theoretical description. In order to find this measure, it is natural to try to explore the presence of a small parameter, viscosity (or inverse Reynolds number). Neglecting viscosity, one obtains the ideal fluid flow, which is a Hamiltonian system. Then, assuming ergodicity, the statistical properties of the flow can be found. Viscosity should deform these properties. It can be taken into account *a posteriori*, by imposing on the ideal fluid flow some constraints, like smoothness, short wave cutoff, boundary constraints, etc. It might happen that this approach leads nowhere, and no modification of the ergodic Hamiltonian measure can approach the measure of the attractor. Such a conclusion, however, cannot be made before this course is followed, and the ergodic Hamiltonian measures are found and checked against experimental data. In any case, determination of the ergodic measure of an ideal fluid is an interesting task in its own right. A certain amount of progress has been made, in some respects only partially, for the case of two-dimensional flows in closed domains (see Refs. [1–9] and references therein). It is not a very interesting case in terms of applications, but it is the simplest one. In this paper, a probabilistic measure for three-dimensional flow is proposed.

The concepts of statistical mechanics cannot be applied directly to fluid motion because fluid possesses an infinite number of degrees of freedom, and it is not clear what should be an analogy of such notions as ergodicity, phase volume, etc. A natural approach would be to truncate the Euler equations and obtain a finite-dimensional system, most desirably a Hamiltonian one. Then, assuming ergodicity of the finite-dimensional system, one can find its thermodynamical and probabilistic characteristics and consider the limit $N\to\infty$. We will use this approach.

There is an important decision to make at the very beginning of the study: Truncations eliminate some properties of the Euler equations, and different truncations eliminate different properties. These differences may persist in the limit $N\to\infty$ and yield different limit results. We have to decide which properties can be sacrificed without detriment to approximation of the attractor measure. The degree of understanding of fluid dynamics that is necessary for such a decision, does not exist at present. In addition, the situation is considerably complicated by the existence of an infinite number of integrals of motion, besides the energy. They are circulations of velocity over closed fluid contours. In infinite-dimensional phase space motion occurs on the cross section of the energy surface by an infinite number of surfaces, which are the images of all other integrals. The cross sections are some infinite-dimensional sheets on the energy surface. A truncation made without necessary precautions violates the extra integrals of motion. It is not clear how important it is to respect the extra integrals of motion since viscosity destroys these integrals. If the extra integrals are changing on time scales that are much larger than the characteristic time of mixing, then these integrals cannot be neglected. We consider the truncations in which all extra integrals of fluid motion are taken into account automatically. One of the key points in the consideration is a note that the dynamics of an ideal incompressible fluid can be split into two parts: the dynamics of the vortex lines and the dynamics of the fluid particles on the vortex lines. The vortex line is a curve in three-dimensional space the tangent vector of which is proportional to the vorticity vector at each point. The choice of parameter on the curve is not important: the curves with different parametrizations are identified. It turns out that the motion of the fluid particles along the vortex lines does not affect the dynamics of the vortex lines. Therefore, the dynamics of the vortex lines can be considered independently and has an intrinsic meaning. The dynamics of fluid particles on the vortex lines is determined uniquely by the motion of the vortex lines from the incompressibility condition.

The dynamics of the vortex lines possesses a remarkable feature: it is Hamiltonian. The corresponding variational principle is offered in this paper. This is a generalization of the variational principle of Ref. [9] for two-dimensional flows. The variational principle presents the equations of fluid dynamics in the form where the extra integrals, circulation of velocity over fluid contours, are eliminated.

For two-dimensional flows the statistical independence of positions of point vortices yields the so-called mean-field theory. Accepting that, for three-dimensional flows, positions of vortex lines are statistically independent, we obtain the following probability measure: probability P of a vortex loop carrying the vorticity $\mathring{\omega}$ to be within a small tube surrounding contour γ is

$$P=\text{const}\times e^{-\beta\mathring{\omega}\int_{\gamma}\overline{\psi_i}dx^i}. \tag{1.1}$$

Here $\overline{\psi_i}$ are the components of the average stream function vector (Latin indices run values 1,2,3, summation over repeated indices is implied), and β plays the role of inverse temperature. Formula (1.1) indicates that the motion of various pieces of the vortex line is also ''almost'' statistically independent: if contour γ is composed of two contours γ_1 and γ_2, the corresponding probabilities are multiplied.

It is shown (Sec. XI) that the probabilistic measure (1.1) yields the probabilistic measure for two-dimensional flows: the probability density function for positions r of the particle carrying vorticity $\mathring{\omega}$ is

$$f(r)=\text{const}\times e^{-\beta\mathring{\omega}\overline{\psi(x)}}, \tag{1.2}$$

where $\overline{\psi}$ is the averaged stream function of the two-dimensional flow. This fact may be considered as a partial explanation of why two-dimensional theory of point vortices predicts [10] turbulent velocity profiles in Couette and Poiseuille flows in spite of the essential three dimensionality of these flows.

Measure (1.1) is used to obtain the equation for the averaged stream function of pipe flow. Previously, the statistical mechanics of vortex lines has been considered by Chorin in a series of papers [11–15] and monograph [8]. Chorin assumed the Gibbs measure,

$$P=\text{const}\times e^{-\beta H}, \tag{1.3}$$

where H is the energy of the vortex line,

$$H=\tfrac{1}{2}\gamma^2 ⨏⨏ \frac{d\eta d\eta'}{|x(\eta)-x(\eta')|} \tag{1.4}$$

(the sign $⨏$ means that some regularization is done to eliminate the divergence of the integral). Measure (1.3),(1.4) was used to obtain the energy spectrum in the inertial range.

Measures (1.1) and (1.3),(1.4) are different. Measure (1.3),(1.4) describes the statistics of a single vortex line in unbounded space if the major contribution to the energy of the flow is the energy of the vortex line under consideration. Measure (1.1), as will be seen from the derivation, represents another physical situation. It describes the statistics of the vortex lines when contribution of any single vortex line to energy is assumed to be negligible in comparison with the energy of the whole ensemble: the intensity of each vortex line tends to zero when the number of vortex lines tends to infinity and only the energy of the ensembles matters.

Other issues addressed in this paper are an explanation of why statistical theory based on dynamics of ideal fluid may describe the turbulent flows of viscous fluids in wall-bounded domains (Appendix C) and the determination of symmetries of the action functional and the corresponding integrals of motion of the vortex line dynamics (Appendix B).

The paper is organized as follows. In the next section the basic equations of an ideal fluid flow are recalled. Then a variational principle for two-dimensional flows is formulated. The expression for the kinetic energy of three-dimensional (3D) flows is derived in Sec. IV. The variational principle for vortex lines is formulated and discussed in Sec. V. In Sec. VI, it is shown how to find the motion of the particles on the vortex lines if the motion of the vortex lines is known. In Sec. VII, the probability measure for two-dimensional flows derived in Ref. [9] from the ergodic hypothesis is obtained by means of the principle of maximum entropy. The approach of Sec. VI is generalized to three-dimensional flows in Sec. VIII. After derivation of some auxiliary relations in Sec. IX, the two-dimensional measure (1.2) is obtained from (1.1) in Sec. X. The averaged equation for the stream function of pipe flow is developed in Sec. XII. This is followed by Appendix A, containing the derivation of the action functional variation, and the above-mentioned Appendixes B and C.

Some of these issues will be discussed also in [16].

II. EQUATIONS OF FLUID DYNAMICS

There are many different forms of the system of equations governing the dynamics of an ideal incompressible fluid flow. In Cartesian Eulerian coordinates, x^i $(i=1,2,3)$, an inertial observer's frame, the system consists of momentum equations (the Euler equations),

$$\rho\left(\frac{\partial v_i(t,x)}{\partial t}+v^j\frac{\partial v_i}{\partial x^j}\right)=-\frac{\partial p}{\partial x^i}, \tag{2.1}$$

and the incompressibility condition,

$$\frac{\partial v^i}{\partial x^i}=0. \tag{2.2}$$

Here v^i and p are velocity components and pressure, correspondingly, and the mass density of fluid, ρ, is assumed to be a constant.

Equations (2.1) and (2.2) form a set of four equations for four unknown functions v^i and p.

Equations (2.1) and (2.2) admit a continuum set of integrals. To find these integrals it is convenient to rewrite Eqs. (2.1) and (2.2) in terms of Lagrangian coordinates ξ^a, (a

$=1,2,3)$. In Lagrangian coordinates, the key required functions are the particle trajectories,

$$x^i = x^i(t,\xi^a),$$

while the velocity components v^i, by definition, are

$$v^i = \frac{\partial x^i(t,\xi^a)}{\partial t}. \tag{2.3}$$

We use a convention, that, for any function φ, derivative $\partial\varphi(t,\xi)/\partial t$ means the time derivative for fixed Lagrangian coordinates while $\partial\varphi(t,x)/\partial t$ means the time derivative for fixed Eulerian coordinates. The sets of coordinates x^i and ξ^a are denoted by x and ξ, correspondingly, if it cannot create misinterpretation.

If the velocity were known as a function of Eulerian coordinates and time then Eq. (2.3) becomes a system of ordinary differential equations to find particle trajectories:

$$\frac{\partial x^i(t,\xi)}{\partial t} = v^j(t,x^j(t,\xi)). \tag{2.4}$$

Equations of fluid dynamics can be written as equations for functions $x^i(t,\xi)$ and $p(t,\xi)$. Indeed, since

$$\frac{\partial v_i(t,x)}{\partial t} + v^j\,\frac{\partial v_i}{\partial x^j} \equiv \frac{\partial^2 x_i(t,\xi)}{\partial t^2},$$

the Euler Eqs. (2.1) can be written as

$$\rho\,\frac{\partial^2 x_i(t,\xi)}{\partial t^2} = -\frac{\partial p(t,\xi)}{\partial \xi^a}\,\frac{\partial \xi^a}{\partial x^i}. \tag{2.5}$$

Here one should understand under $\partial\xi^a/\partial x^i$ the components of the matrix that is inverse to the matrix $\|\partial x^i/\partial\xi^a\|$. For given p, Eqs. (2.5) form a system equations of second order for $x(t,\xi)$.

The incompressibility condition can be formulated in terms of functions $x(t,\xi)$ as the conservation of the determinant of matrix $\|\partial x/\partial\xi\|$ at each particle,

$$\det\left\|\frac{\partial x}{\partial \xi}\right\| = \sqrt{\mathring{g}(\xi)}, \tag{2.6}$$

where $\mathring{g}(\xi)$ is the determinant of the metric tensor in Lagrangian coordinates. The equivalence of Eqs. (2.2) and (2.6) can be derived from the identity

$$\left.\frac{\partial}{\partial t}\right|_{\xi=\text{const}} \det\left\|\frac{\partial x}{\partial \xi}\right\| = \det\left\|\frac{\partial x}{\partial \xi}\right\|\,\frac{\partial v^i}{\partial x^i}.$$

Equations (2.5) and (2.6) form a system of four equations for four required functions, $x^i(t,\xi)$ and $p(t,\xi)$.

Another remarkable form of the Euler equations (2.5) is obtained if these equations are written for the covariant components of velocity in Lagrangian coordinates,

$$v_a = \frac{\partial x^i}{\partial \xi^a}\,\frac{\partial x_i}{\partial t}. \tag{2.7}$$

Contracting Eq. (2.5) with $\partial x^i/\partial\xi^a$ and differentiating by parts we obtain

$$\rho\,\frac{\partial v_a(t,\xi)}{\partial t} = -\frac{\partial}{\partial \xi^a}\,(p - \tfrac{1}{2}\rho v^2). \tag{2.8}$$

Here the squared absolute value of velocity is denoted by v^2,

$$v^2 = \frac{\partial x^i(t,\xi)}{\partial t}\,\frac{\partial x_i(t,\xi)}{\partial t}.$$

The system of equations admits the reduction of the order. Indeed, let us define the function $\varphi(t,\xi)$ by the equation

$$\rho\,\frac{\partial \varphi(t,\xi)}{\partial t} = -p + \tfrac{1}{2}\rho v^2. \tag{2.9}$$

For a given motion $x(t,\xi)$ this equation establishes the one-to-one correspondence between p and φ if initial data for φ are provided. Let for definiteness φ be zero initially. Then, from Eqs. (2.8) and (2.9),

$$v_a(t,\xi) = \mathring{v}_a(\xi) + \frac{\partial \varphi(t,\xi)}{\partial \xi^a}, \tag{2.10}$$

where $\mathring{v}_a(\xi)$ are the initial values of velocity. From Eqs. (2.7) and (2.10) we obtain the equations for $x(t,\xi)$,

$$\frac{\partial x^i}{\partial \xi^a}\,\frac{\partial x_i(t,\xi)}{\partial t} = \mathring{v}_a(\xi) + \frac{\partial \varphi(t,\xi)}{\partial \xi^a}, \tag{2.11}$$

where function φ should be chosen in such a way that the incompressibility condition (2.6) holds.

After the system of Eqs. (2.11) and (2.6) is solved, pressure can be found from Eq. (2.9).

The time-dependent potential part of velocity (2.10) can be eliminated by differentiation of Eq. (2.10) with respect to b and alternating a and b. We have

$$\frac{\partial}{\partial t}\,v_{[a,b]}(t,\xi) = 0, \tag{2.12}$$

where

$$v_{[a,b]} \equiv \frac{1}{2}\left(\frac{\partial v_a}{\partial \xi^b} - \frac{\partial v_b}{\partial \xi^a}\right).$$

Antisymmetric tensor $v_{[a,b]}$ is in one-to-one correspondence with the vector,

$$\omega^c \equiv \frac{1}{\sqrt{\mathring{g}}}\,e^{abc} v_{[a,b]}, \quad v_{[a,b]} = \frac{1}{\sqrt{\mathring{g}}}\,e_{abc}\omega^c \tag{2.13}$$

Here $e^{abc} = e_{abc}$, are the components of the Levi-Civita symbol.

The vector with contravariant components ω^a is called the vorticity vector. In accordance with Eq. (2.12), at each fluid particle, $\omega^a(\xi)$ do not depend on time and are equal to their initial values, which are denoted by $\mathring{\omega}^a(\xi)$.

In Eulerian coordinates the vorticity vector has the components

$$\omega^i = e^{ijk} \frac{\partial v_j}{\partial x^k} \tag{2.14}$$

and, due to the law of transformation of vector components,

$$\omega^i = \frac{\partial x^i}{\partial \xi^a} \mathring{\omega}^a. \tag{2.15}$$

Inversely,

$$\mathring{\omega}^a = \frac{\partial \xi^a}{\partial x^i} \omega^i. \tag{2.16}$$

Note that the covariant components of vorticity,

$$\omega_a = \frac{\partial x^i}{\partial \xi^a} \omega_i,$$

in general, are time dependent.

Conservation of vorticity is often formulated in another form: for any closed fluid contour Γ circulation of velocity v_i, $\int_\Gamma v_i dx^i$, does not depend on time. This statement is equivalent to conservation of $\mathring{\omega}^a$.

A form of Eq. (2.15) that does not refer to Lagrangian coordinates can be obtained by differentiation (2.15) with respect to time for fixed Lagrangian coordinates:

$$\frac{\partial \omega^i(t,\xi)}{\partial t} = \frac{\partial \omega^i(t,x)}{\partial t} + v^k \frac{\partial \omega^i}{\partial x^k} = \frac{\partial v^i}{\partial \xi^a} \mathring{\omega}^a.$$

Eliminating $\mathring{\omega}^a$ in the last relation by means of Eq. (2.16) we have

$$\frac{\partial \omega^i(t,x)}{\partial t} + v^k \frac{\partial \omega^i}{\partial x^k} = \frac{\partial v^i}{\partial x^k} \omega^k. \tag{2.17}$$

This equation can also be obtained by applying the Rot operator to the Euler equations (2.1). As follows from the derivation, this equation is equivalent to conservation of the contravariant components of vorticity in Lagrangian coordinates, $\mathring{\omega}^a$.

A certain difficulty in solution of the Euler equations in the form (2.11) relates to necessity to satisfy the incompressibility condition. Fortunately, there is a way to get around this difficulty. The point is that any incompressible velocity field is completely determined by the vorticity field, and there is the integral relation,

$$v^i(t,x) = \int_V R^i_j(x,x') \omega^j(t,x') d^3x'. \tag{2.18}$$

The kernel $R^i_j(x,x')$, depends on the geometry of region V. This relation is discussed in more detail in Sec. IV. From Eqs. (2.18), (2.15), and (2.6) one obtains a system of integrodifferential equations for fluid particle trajectories

$$\frac{\partial x^i(t,\xi)}{\partial t} = \int_V R^i_j(x(t,\xi),x(t,\xi')) \frac{\partial x^j(t,\xi')}{\partial \xi'^a} \mathring{\omega}^a(\xi') d^3\xi'. \tag{2.19}$$

In this form, the incompressibility and the conservation of vorticity become the built-in properties of the right-hand side of the equation. Unfortunately, there is an unpleasant feature of Eqs. (2.19): in contrast to the Euler equations (2.1), they are not Hamiltonian. It turns out, however, that this system can be split into two parts: one subsystem is closed and Hamiltonian and describes the dynamics of the vortex lines while another one determines the motion of particles on the vortex lines by a known solution of the first subsystem. We consider first the 2D case when the motion of particles along the vortex lines is absent, and the system (2.19) is Hamiltonian.

III. 2D MOTION

A. Equations of 2D motion of an ideal incompressible fluid

For 2D motion one of the Eulerian coordinates, say, x^3, is identically equal to ξ^3, while two others, $x^\alpha,(\alpha=1,2)$, are functions of t and $\xi^\mu(\mu=1,2)$ only:

$$x^\alpha = x^\alpha(t,\xi^\mu), \quad x^3 \equiv \xi^3. \tag{3.1}$$

Here and in what follows Greek indices run values 1,2. For 2D motion $v^3 \equiv 0$ while v^α components obey the incompressibility condition,

$$\frac{\partial v^\alpha}{\partial x^\alpha} = 0.$$

The latter means that a function $\psi(t,x)$ exists such that

$$v^\alpha = e^{\alpha\beta} \frac{\partial \psi(t,x)}{\partial x^\beta}.$$

Here $e^{\alpha\beta}$ are the components of the 2D Levi-Chivita symbol, $e^{11}=e^{22}=0$, $e^{12}=-e^{21}=1$.

The vorticity vector has the only nonzero component, $\omega^3 \equiv \omega(t,x)$, and

$$-\Delta \psi = \omega. \tag{3.2}$$

At the boundary the ''no-penetration-detachment'' condition is accepted, $v^\alpha n_\alpha = 0$. Assuming also, for simplicity, that region V is simply connected, we have without loss of generality

$$\psi|_{\partial V} = 0. \tag{3.3}$$

Equations (3.2) and (3.3) have the solution

$$\psi(t,x) = \int_V G(x,x') \omega(t,x') d^2x', \tag{3.4}$$

where $G(r,r')$ is the Green's function of region V determined by the boundary-value problem (Δ_r is the Laplace operator in r variables):

$$\Delta_r G(r,r') = -\delta(r-r') \quad \text{in } V, \quad G(r,r')=0 \quad \text{if } r \in \partial V. \tag{3.5}$$

The components of velocity are

$$v_1(t,x) = \frac{\partial}{\partial x_2} \int_V G(x,x') \omega(t,x') d^2x', \tag{3.6}$$

$$v_2(t,x) = -\frac{\partial}{\partial x_1}\int_V G(x,x')\omega(t,x')d^2x'.$$

Equations (3.6) form a 2D version of the general equations (2.18).

In accordance with Eqs. (3.1) and (2.16) conservation of vorticity means that

$$\omega(t,x^\alpha(t,\xi^\mu)) = \mathring{\omega}(\xi^\mu) \tag{3.7}$$

or

$$\omega(t,x^\alpha) = \mathring{\omega}(\xi^\mu(t,x^\alpha)). \tag{3.8}$$

Hence, Eqs. (2.19) in the 2D case take the form

$$\frac{dx(t,\xi)}{dt} = \int_V \left.\frac{\partial G(r,r(t,\xi'))}{\partial y}\right|_{r=r(t,\xi)} \mathring{\omega}(\xi')d^2\xi',$$

$$\frac{dy(t,\xi)}{dt} = -\int_V \left.\frac{\partial G(r,r(t,\xi'))}{\partial x}\right|_{r=r(t,\xi)} \mathring{\omega}(\xi')d^2\xi'. \tag{3.9}$$

Here $x_1 \equiv x$, $x_2 \equiv y$, and couple (x,y) is denoted by r.

Equations (3.9) determine the dynamics of an ideal incompressible fluid in a bounded simply connected domain. The flow is specified by the prescribed initial vorticity. The incompressibility condition and conservation of vorticity have been used in construction of Eq. (3.9) and form the properties that can be easily respected in truncations. It turns out that dynamical equations are Hamiltonian and follow from the variational principle to which we proceed.

B. Variational principle

Consider the following functional of the position vector $r(t,\xi)$:

$$I(r) = \int_{t_0}^{t_1} dt\left[\int_V \mathring{\omega}(\xi)y(t,\xi)\frac{dx(t,\xi)}{dt}d^2\xi - K\right], \tag{3.10}$$

$$K = \frac{1}{2}\int_V\int_V G(r(t,\xi),r(t,\xi'))\mathring{\omega}(\xi)\mathring{\omega}(\xi')d^2\xi d^2\xi'. \tag{3.11}$$

Here $d^2\xi \equiv d\xi^1 d\xi^2$.

It can be checked by inspection that the stationary points of functional (3.10) are solutions of the equations of fluid dynamics (3.9). To eliminate extra terms in variation of the functional (3.10) appeared at $t=t_0,t_1$ after integration by parts, one can assume that the x coordinates of all particles are prescribed at $t=t_0,t_1$.

Some features of this variational principle are worth noting. First, the first term in Eq. (3.10) has the form of standard ''shortened'' action functional in classical mechanics $\int p\dot{q}\,dt$ if x and y coordinates of particles are identified with generalized coordinate and momentum while summation over degrees of freedom corresponds to summation over fluid particles with the measure $\mathring{\omega}(\xi)d^2\xi$. The second term in Eq. (3.10) is the Hamiltonian of the system, which is the total kinetic energy of the fluid. This justifies the term ''action functional'' for the functional (3.10). Second, in dynamical equations in the form (3.9) the integrals of fluid motion additional to energy are eliminated (see Appendix B for more detail). Third, the number of degrees of freedom is decreased significantly: only particles carrying nonzero vorticity are taken into consideration. Particles with zero vorticity influence the dynamics through the Green's function, which is determined completely by geometry of the region V. Fourth, in contrast to the usual Hamiltonian variational principle in fluid dynamics, (see, for example, [17]) the admissible functions $r(t,\xi)$ are arbitrary and should not satisfy the incompressibility condition, $\det\|\partial r/\partial\xi\| = 1$: each stationary point of the functional (3.10) obeys this condition automatically.

C. Point vortex truncation

Action functional (3.10) determines a Hamiltonian system of an infinite number of particles.

A natural finite-dimensional truncation of the continuum would be to keep only a finite number N of particles, $\xi_1,\ldots,\xi_N$. Then motion of continuum is characterized by N functions $r_1(t) \equiv r(t,\xi_1),\ldots,r_N(t) = r(t,\xi_N)$.

From now on we have to use indices of three various natures: indices corresponding to the projections on Eulerian and Lagrangian coordinates, and indices numbering the selected particles (point vortices) or vortex lines. In 3D for these purposes we use three groups of Latin indices, (i,j,k,m,n), (a,b,c), and (p,q,s,t) correspondingly; in 2D, Eulerian and Lagrangian indices are denoted by Greek letters (α,β,γ) and (μ,ν,λ), respectively.

Approximating $\mathring{\omega}(\xi)$ by δ functions:

$$\mathring{\omega}(\xi) = \sum_{s=1}^{N} \gamma_s\delta(\xi - \xi_s), \tag{3.12}$$

one obtains the action functional of the point vortex approximation,

$$I(r_i) = \int_{t_0}^{t_1}\left[\sum_s \gamma_s y_s \dot{x}_s - H(r)\right]dt, \tag{3.13}$$

$$H(r) = \frac{1}{2}\sum_{s,t} G(r_s,r_t)\gamma_s\gamma_t. \tag{3.14}$$

In the expression for $G(r_s,r_s)$ the leading (infinite) term can be dropped since it is independent of motion. Then the functional (3.13) is the action functional in the theory of point vortices.

IV. KINETIC ENERGY OF THREE-DIMENSIONAL FLOWS

In order to extend the variational principle discussed to three-dimensional flows, one has to obtain, first of all, the expression for kinetic energy similar to Eq. (3.11). Let us show that the total kinetic energy K can be presented in the form

$$K=\frac{1}{2}\int_V\int_V G_{ij}(x,x')\omega^i(t,x)\omega^j(t,x')d^3x d^3x', \quad (4.1)$$

where $G_{ij}(x,x')$ is some two-point tensor field determined by the geometry of the container only.

Consider the well-known kinematical problem: find the velocity field generated by a given vorticity. The vorticity field obeys the constraint

$$\frac{\partial\omega^i(x)}{\partial x^i}=0. \quad (4.2)$$

For any vorticity field satisfying Eq. (4.2) the velocity field $v_i(x)$ is determined from the system of equations

$$e^{ijk}\frac{\partial v_k(x)}{\partial x^j}=\omega^i(x), \quad \frac{\partial v^i}{\partial x^i}=0, \quad v^i n_i=0 \quad \text{at } \partial V. \quad (4.3)$$

Here n_i are the components of the unit outward normal vector at ∂V. It is easy to see that the problem (4.3) has a unique solution. This solution can be written in the form

$$v^i(x)=\int_V R^i_j(x,x')\omega^j(x')d^3x'. \quad (4.4)$$

The kernel $R^i_j(x,x')$ is not unique due to the condition (4.2): one can change R^i_j by adding the tensor $\partial R^i(x,x')/\partial x'^j$, where $R^i(x,x')$ is an arbitrary vector field vanishing if $x'\in\partial V$.

Let us introduce the stream function vector $\psi_i(x)$ by the system of equations similar to Eq. (4.3):

$$e^{ijk}\frac{\partial\psi_k(x)}{\partial x^j}=v^i(x), \quad \frac{\partial\psi^i(x)}{\partial x^i}=0, \quad \psi^i n_i=0 \quad \text{at } \partial V. \quad (4.5)$$

The stream function vector determined by Eq. (4.5) is unique. Similarly to Eq. (4.4) one can write

$$\psi^i(x)=\int_V R^i_j(x,x')v^i(x')d^3x'. \quad (4.6)$$

The difference between the problems (4.3) and (4.5) is that velocity field is constrained by the ‘‘no-penetration-detachment’’ condition

$$v^i n_i=0 \quad \text{at } \partial V. \quad (4.7)$$

To make kinematical problems (4.3) and (4.5) completely identical, we assume that vorticity satisfies the similar condition

$$\omega^i n_i=0 \quad \text{at } \partial V. \quad (4.8)$$

This condition helps to simplify some further relations. Physically, this condition is sensible: since the ideal fluid flow is considered as an approximation of viscous flow, and for viscous flows with no-slip boundary conditions equation (4.8) holds, it is natural to accept that an ideal fluid flow inherits the boundary condition (4.8).

To write down Eq. (4.8) in terms of velocity we introduce surface curvilinear coordinates ζ^α on ∂V and denote projections on these coordinates by Greek indices; they run on values 1,2. In accordance with the first equation (4.3), Eq. (4.8) takes the form

$$e^{\alpha\beta}\frac{\partial v_\alpha}{\partial\zeta^\beta}=0. \quad (4.9)$$

Equation (4.9) means that the tangent components of velocity are some potential functions at ∂V,

$$v_\alpha=\frac{\partial\chi}{\partial\zeta^\alpha}, \quad (4.10)$$

where χ is an arbitrary function of surface coordinates.

As soon as we constrained the admissible vorticity fields by the boundary condition (4.8), the kernel $R_{ij}(x,x')$ gets the property

$$\int_V\int_V[R_{ij}(x,x')-R_{ji}(x',x)]\underset{1}{\omega^i}(x)\underset{2}{\omega^j}(x')d^3x d^3x'=0 \quad (4.11)$$

for any two divergence-free vector fields $\underset{1}{\omega^i}$ and $\underset{2}{\omega^i}$.

To prove Eq. (4.11), denote by $\underset{1}{v^i}$ and $\underset{2}{v^i}$ the velocity fields determined by $\underset{1}{\omega^i}$ and $\underset{2}{\omega^i}$ from Eq. (4.3), and consider the integral

$$B=\int_V \underset{2}{v_i}\underset{1}{\omega^i}d^3x=\int_V\int_V R_{ij}(x,x')\underset{2}{\omega^j}(x')\underset{1}{\omega^i}(x)d^3x'd^3x. \quad (4.12)$$

Integrating by parts we have

$$B=\int_V \underset{2}{v_i}e^{ijk}\frac{\partial \underset{1}{v_k}}{\partial x^j}d^3x=\int_{\partial V}\underset{2}{v_i}e^{ijk}n_j\underset{1}{v_k}d^2x-\int_V \underset{1}{v_k}e^{ijk}\frac{\partial \underset{2}{v_i}}{\partial x^j}d^3x.$$

Since, in accordance with Eq. (4.10),

$$\int_{\partial V}\underset{2}{v_i}e^{ijk}n_j\underset{1}{v_k}d^2x=-\int \underset{2}{v_\alpha}e^{\alpha\beta}\underset{1}{v_\beta}d^2\zeta=-\int\frac{\partial}{\partial\zeta^\alpha}\left(\chi_2 e^{\alpha\beta}\frac{\partial\chi_1}{\partial\zeta^\beta}\right)d^2\zeta=0,$$

we obtain

$$B=\int_V \underset{1}{v_i}\underset{2}{\omega^i}d^3x=\int_V\int_V R_{ij}(x,x')\underset{1}{\omega^j}(x')\underset{2}{\omega^i}(x)d^3x d^3x'. \quad (4.13)$$

Equation (4.11) follows from Eqs. (4.12) and (4.13).

Composition of formulas (4.4) and (4.6) yields the expression of stream function vector ψ^i in terms of vorticity,

$$\psi_i(x)=\int_V G_{ij}(x,x')\omega^j(x')d^3x', \tag{4.14}$$

where

$$G_{ij}(x,x')=\int_V R_{ik}(x,\widetilde{x})R_j^k(\widetilde{x},x')d^3\widetilde{x}. \tag{4.15}$$

Let us show that tensor G_{ij} is the kernel in the expression for kinetic energy (4.1). We have

$$2K=\int_V v_i v^i d^3x=\int_V v_i e^{ijk}\frac{\partial\psi_k}{\partial x^j}d^3x=\int_{\partial V} v_i e^{ijk}n_j\psi_k d^2\zeta$$

$$-\int_V \frac{\partial v_i}{\partial x_j}e^{ijk}\psi_k d^3x=-\int_{\partial V} v_\alpha e^{\alpha\beta}\psi_\beta d^2\zeta$$

$$+\int_V \psi_i\omega^i d^3x. \tag{4.16}$$

In accordance with Eqs. (4.7) and (4.8), v_α and ψ_α are some potential vectors at ∂V, and the first integral in Eq. (4.16) vanishes. Thus

$$K=\frac{1}{2}\int_V \psi_i\omega^i d^3x \tag{4.17}$$

and Eq. (4.1) follows from Eqs. (4.14) and (4.17).

Note that an additional contribution in Eq. (4.17) appears if $\omega_n\equiv\omega^i n_i\neq 0$ at ∂V. If χ is the surface potential for $\psi_\beta=\chi_{,\beta}$ (recall that the condition $v^i n_i=0$ is accepted), then this contribution is

$$\int_{\partial V}\chi\omega_n d^2x.$$

It vanishes for $\omega_n=0$.

In general, tensor $G_{ij}(x,x')$ is not symmetric. One can introduce symmetric tensor $\widetilde{G}_{ij}(x,x')$ by the relation

$$\widetilde{G}_{ij}(x,x')=\int_V R_{mi}(\widetilde{x},x)R^{m}{}_{j}(\widetilde{x},x')d^3\widetilde{x}. \tag{4.18}$$

The kinetic energy can be written in terms of $\widetilde{G}_{ij}(x,x')$:

$$K=\frac{1}{2}\int_V v_i v^i d^3x=\frac{1}{2}\int_V\int_V \widetilde{G}_{ij}(x,x')\omega^i(x)\omega^j(x')d^3x d^3x'. \tag{4.19}$$

Expressions (4.19) and (4.1) coincide due to identity (4.11) (this identity should be applied to the vectors

$$\underset{1}{\omega}^i=\omega^i \quad\text{and}\quad \underset{2}{\omega}^i=\int R^i_j(x,x')\omega^j(x')d^3x',$$

which are both divergence free).

V. VARIATIONAL PRINCIPLE FOR VORTEX LINES

A. Vortex lines

The vorticity vector field determines a family of vortex lines. Since vorticity components $\mathring{\omega}^a$ are constant in Lagrangian coordinates, it is worthwhile to consider the fluid lines, $\xi^a=\xi^a(\sigma)$, for which the tangent vector $d\xi^a/d\sigma$ is proportional to $\mathring{\omega}^a$. The Lagrangian coordinate system with one of coordinate lines, say, ξ^3, directed along the vortex lines plays a distinct role. We shall call it the vortex line coordinate system. In this coordinate system only one contravariant component of vorticity, $\mathring{\omega}^3$, is not zero. In fact, the existence of the vortex line coordinate system is an assumption that puts some constraints on the initial vorticity field. This assumption, however, does not seem physically restricting.

In accordance with Eq. (2.13), the vorticity vector is divergence free:

$$\frac{\partial}{\partial\xi^c}\sqrt{\mathring{g}}\mathring{\omega}^c=0. \tag{5.1}$$

Therefore the quantity $\mathring{\omega}=\sqrt{\mathring{g}}\mathring{\omega}^3$ does not depend on ξ^3. Function $\mathring{\omega}(\xi^1,\xi^2)$ plays the role of the intensity of the vortex lines and is similar to function $\mathring{\omega}$ of two-dimensional flows. The only difference is that for two-dimensional flow the coordinates ξ^1,ξ^2 can always be chosen to be Cartesian while for three-dimensional flows Lagrangian coordinates are, in general, curvilinear.

In the vortex line coordinates, a parameter along the vortex lines is denoted by η. The formula for vortex intensity, $\mathring{\omega}=\sqrt{\mathring{g}}\mathring{\omega}^3$, has the same form for all choices of the parameter η: for any other parameter $\eta'=\eta'(\eta,\xi^1,\xi^2)$, the third vorticity component gets the factor $\partial\eta/\partial\eta'$, $\mathring{\omega}^{3'}=\mathring{\omega}^3\partial\eta'/\partial\eta$, while the determinant of the metric tensor gets the factor $(\partial\eta'/\partial\eta)^2$, $\mathring{g}'=\mathring{g}(\partial\eta/\partial\eta')^2$; these factors are canceled in the expression for $\mathring{\omega}$: $\mathring{\omega}=\mathring{\omega}'^3\sqrt{\mathring{g}'}=\mathring{\omega}^3\sqrt{\mathring{g}}$.

In the following, we denote by ξ a couple (ξ^1,ξ^2), so that ξ is a mark of the vortex line, while the whole set of Lagrangian coordinates (ξ^1,ξ^2,ξ^3) is denoted by $\boldsymbol{\xi}$.

B. Kinetic energy as a functional of positions of vortex lines

We presented kinetic energy as a bilinear form of the vorticity vector in Eulerian coordinates. In fact, we need it in Lagrangian coordinates. To perform the transformation, denote by $r(t,\boldsymbol{\xi})$ the position of fluid particle $\boldsymbol{\xi}$ at instant t. The integrals over Eulerian coordinates can be transformed to the integrals over Lagrangian coordinates by means of the following relation: for any function, $\varphi(t,x)$,

$$\int_V\varphi(t,x)d^3x=\int_V\varphi(t,r(t,\boldsymbol{\xi}))\left|\frac{\partial x}{\partial\boldsymbol{\xi}}\right|d^3\boldsymbol{\xi},$$

or, taking into account the incompressibility condition (2.6),

$$\int_V\varphi(t,x)d^3x=\int_V\varphi(t,r(t,\boldsymbol{\xi}))\sqrt{\mathring{g}}d^3\boldsymbol{\xi}. \tag{5.2}$$

From Eqs. (4.1), (5.2), and (2.15) we obtain the kinetic energy as a functional of position vector,

$$K=\frac{1}{2}\int_V\int_V G_{ij}(r(t,\boldsymbol{\xi}),r(t,\boldsymbol{\xi}'))\frac{\partial r^i(t,\boldsymbol{\xi})}{\partial \xi^a}\times\mathring{\omega}^a(\boldsymbol{\xi})\frac{\partial r^j(t,\boldsymbol{\xi}')}{\partial \xi^b}\mathring{\omega}^b(\boldsymbol{\xi}')\sqrt{\mathring{g}(\boldsymbol{\xi})}d^3\boldsymbol{\xi}\sqrt{\mathring{g}(\boldsymbol{\xi}')}d^3\boldsymbol{\xi}'. \tag{5.3}$$

In the vortex line coordinate system the expression (5.3) takes the form

$$K=\frac{1}{2}\int_V\int_V G(\gamma(t,\xi),\gamma(t,\xi'))\mathring{\omega}(\xi)\mathring{\omega}(\xi')d^2\xi d^2\xi', \tag{5.4}$$

where $\gamma(t,\xi)$ is the position of the vortex line ξ at the instant t: $x^k=r^k(t,\eta,\xi)$, and $G(\gamma,\gamma')$ is the functional of vortex line positions γ and γ',

$$G(\gamma,\gamma')\equiv\int\int G_{ij}(r^k(\eta),r'^k(\eta'))\frac{\partial r^i(\eta)}{\partial\eta}\frac{\partial r'^i(\eta')}{\partial\eta'}d\eta d\eta'. \tag{5.5}$$

In Eq. (5.5) $r^k(\eta)$ and $r'^k(\eta')$ are the curves γ and γ'; in Eq. (5.4) $d^2\xi\equiv d\xi^1 d\xi^2$.

Formula (5.4) is an exact 3D analogue of the 2D Eq. (3.11): if the vortex line $\gamma(t,\xi)$ is determined by one point in the plane, $r(t,\xi)$, and $G(\gamma,\gamma')$ is replaced by Green's function $G(r,r')$, Eq. (5.4) is transformed to Eq. (3.11).

The expressions (5.4) and (5.5) show that kinetic energy is invariant with respect to the motion of fluid particles along the vortex lines. Therefore, any variational principle that uses the expressions (5.4) and (5.5) for kinetic energy cannot determine the dynamics of fluid particles on the vortex lines. In the classical Hamilton variational principle the kinetic energy does not have such a symmetry, and the motion of fluid particles is determined uniquely. The action functional of the Hamilton variational principle possesses, however, another rich group of symmetry, the relabeling group. This group generates conservation of vorticity circulations. Writing the energy expression in the form (5.4),(5.5) we eliminated the relabeling group and the corresponding integrals.

C. Variational principle for vortex lines

Consider the following functional of particle positions $r(t,\xi)$:

$$I=\int_{t_0}^{t_1}[A-K]dt,$$

$$A\equiv\int_V\frac{1}{3}e_{ijk}r^i(t,\boldsymbol{\xi})\frac{\partial r^j(t,\boldsymbol{\xi})}{\partial t}\frac{\partial r^k(t,\boldsymbol{\xi})}{\partial\boldsymbol{\xi}^a}\mathring{\omega}^a(\boldsymbol{\xi})d^3\boldsymbol{\xi}, \tag{5.6}$$

where K is the functional (5.4). We are going to show that stationary points of this functional correspond to the motion of an ideal incompressible fluid.

Variation of the functional (5.6) is (see Appendix A)

$$\delta I=\int_{t_0}^{t_1}\int_V e_{ijk}\delta r^i(\dot{r}^j-V^j(r))\omega^k d^3dt, \tag{5.7}$$

where velocity V^j is determined as a functional of position vector:

$$V^j(r)=e^{jlm}\int_V\left.\frac{\partial R_{lk}(r,r(t,\boldsymbol{\xi}'))}{\partial r^m}\right|_{r=r(t,\xi)}\times\frac{\partial r^k(t,\boldsymbol{\xi}')}{\partial\xi'^a}\mathring{\omega}^a(\boldsymbol{\xi}')d^3\boldsymbol{\xi}'. \tag{5.8}$$

Since δr^i are arbitrary, the Euler equations of the variational problem are

$$e_{ijk}\left(\frac{\partial r^j(t,\boldsymbol{\xi})}{\partial t}-V^j(r)\right)\omega^k=0, \tag{5.9}$$

where ω^k should be expressed in terms of position vector by Eq. (2.15) [18]. Symmetry of the action functional with respect to the particle motion along the vortex lines causes the Euler equations (5.9) to be dependent: contraction of Eq. (5.9) with the vorticity vector yields identity.

Equation (5.9) can be rewritten also as

$$\frac{\partial r^i(t,\boldsymbol{\xi})}{\partial t}=V^i(r)+\lambda(t,\boldsymbol{\xi})\omega^i, \tag{5.10}$$

where λ is an arbitrary function of t and $\boldsymbol{\xi}$.

It is clear that the choice of λ determines the particle velocity along the vortex lines and does not affect the motion of the vortex lines themselves. Projection of the velocity to the directions that are normal to the vortex lines coincides with that for ideal fluid flow. Thus, the dynamics of the vortex lines is determined correctly by the variational principle formulated.

The uncertainty of positions of particles on the vortex lines suggests the interpretation of the variational principle for functional (5.6) as a variational principle for the dynamics of vortex lines.

The variational principle has the same advantages as that mentioned with regard to the variational principle for functional (3.10).

Note that functional (5.6) cannot be reduced to the functional (3.10) for two-dimensional motion: it is enough to notice the factor 1/3 in the expression for A [Eq. (5.6)]. The reason is that the condition $\omega^i n_i=0$ at ∂V was used: this condition is violated for two-dimensional flows. Nevertheless, conceptually functional (5.6) may be viewed as a three-dimensional analogue of the functional (3.10).

The variational principle suggests a natural way to truncate fluid dynamics equations. First, the continuum set of vortex lines is replaced by a finite set of vortex lines, $\xi_1,\ldots,\xi_N$. That is similar to the point vortex truncation. Second, the dynamics of each vortex line is sought within an m-parameter family of closed 3D curves,

$$x^i=r^i(s,\alpha^1,\ldots,\alpha^m). \tag{5.11}$$

Here s is the arc length along the curve, and the parameters $\alpha^1,\ldots,\alpha^m$ specify the member of the family. The dependence

$\alpha^1=\alpha^1(t),\ldots,\alpha^m=\alpha^m(t)$ determines the dynamics of the line. The functions (5.11) may depend also on ξ: $x=r(s,\alpha,\xi)$.

So, the dynamics of fluid is described by a finite set of generalized coordinates $\alpha=(\alpha_A^i)$, $i=1,...,m$, $A=1,...,N$; the parameters α_A^i with index A correspond to the Ath vortex line. Plugging the admissible functions (5.11) into the action functional (5.6) and integrating over ξ and s, one obtains

$$A=\sum_{i,A} P_i^A(\alpha)\dot{\alpha}_A^i, \quad K=K(\alpha), \tag{5.12}$$

where

$$P_i^A=\gamma_A\int \frac{1}{3} e_{mjk}r^m(s,\alpha,\xi_A)\frac{\partial r^j(s,\alpha,\xi_A)}{\partial \alpha_A^i}\frac{\partial r^j(s,\alpha,\xi_A)}{\partial s}ds$$

and γ_A is the intensity of the Ath vortex line, $\gamma_A=\mathring{\omega}(\xi_A)\Delta$, and Δ is the 2D blob size (or area of the vortex tube cross section). So, the truncated system is Hamiltonian and has the Lagrange function,

$$L=\sum_{i,A} P_i^A(\alpha)\dot{\alpha}_A^i-K(\alpha). \tag{5.13}$$

Assuming that the motion is ergodic in the α variables one can study the statistical properties of the system (5.13). We emphasize that one needs the ergodicity of the motion of vortex lines only: one may not expect ergodicity if the motion of the fluid particles along the vortex lines is taken into account. (The latter depends on the value of the velocity circulation over the vortex lines. If it is zero, as one may assume for the Couette and Poiseuille flows, then the motion of particles is not ergodic. For nonzero circulations it may be ergodic.)

Note that the Lagrange function of the point vortex dynamics also has the form (5.13).

Denote temporarily all coordinates α by $x=(x^1,\ldots,x^n)$, $n=mN$. The Lagrange function has the form

$$L(x,\dot{x})=\sum_{i=1}^{n} P_i(x)\dot{x}^i-H(x). \tag{5.14}$$

This Lagrange function does not have the standard form of Hamiltonian mechanics,

$$L=\sum p_i\dot{q}^i-H(p,q).$$

The standard form corresponds to even $n=2m$, and functions $P_i(x)$ of the form

$$P_1(x)=x^{m+1},\ldots, \quad P_m(x)=x^{2m},$$

$$P_{m+1}=0,..., \quad P_{2m}=0.$$

In this case we identify $x^1,\ldots,x^m$ with $q^1,\ldots,q^m$ and $x^{m+1},\ldots,x^{2m}$ with $p^1,\ldots,p^m$, respectively. In the general case, for even $n=2m$, the form $\Sigma_{i=1}^n P_i(x)dx^i$ can be transformed to $\Sigma_{i=1}^m p_i\,dq^i$ by a coordinate transformation (at least, locally). However, to find such a transformation is not an easy task, and we are obliged to consider the truncated dynamics of the vortex lines in x coordinates. In x coordinates, the basic relations of classical statistical mechanics should be slightly modified (see [16]). Fortunately, the statements that follow will not be affected.

VI. MOTION OF PARTICLES ON VORTEX LINE

A remarkable property of vortex lines is that the dynamics of the vortex lines determines the dynamics of the fluid completely. More precisely, let

$$x^k=r^k(t,\eta,\xi) \tag{6.1}$$

be a parametric form of equations of vortex line ξ. If the motion of all vortex lines is known then the motion of particles still has some arbitrariness due to possible motion along the vortex lines. To describe the arbitrariness let us refer each vortex line at instant t to the arc length s along the vortex line at this instant,

$$x^k=\mathring{r}^k(t,s,\xi). \tag{6.2}$$

The functions $\mathring{r}^k(t,s,\xi)$ are determined by the positions of the vortex lines. The length of the vortex line $\xi, l(t,\xi)$, depends, in general, on t, and the arc length s in Eq. (6.2) changes within the limits

$$0\leqslant s\leqslant l(t,\xi).$$

Any motion of particles along the vortex line ξ is described by the function

$$s=s(t,\eta,\xi).$$

The functions $r^k(t,\eta,\xi)$ relate to the positions of the vortex lines $\mathring{r}^k(t,s,\xi)$ by the equations

$$r^k(t,\eta,\xi)=\mathring{r}^k(t,s(t,\eta,\xi),\xi).$$

It turns out that motion along vortex lines, $s(t,\eta,\xi)$, is determined by the motion of vortex lines due to the incompressibility condition. To show that let us denote the determinant of the matrix with the rows $\partial\mathring{r}^k/\partial\xi^1,\partial\mathring{r}^k/\partial\xi^2,\partial\mathring{r}^k/\partial s$ by

$$\det\left\|\frac{\partial\mathring{r}}{\partial\xi},\frac{\partial\mathring{r}}{\partial s}\right\|.$$

The similar notation is used for the determinant of the matrix with the rows $\partial r^k/\partial\xi^1,\partial r^k/\partial\xi^2,\partial r^k/\partial\eta$:

$$\det\left\|\frac{\partial r}{\partial\xi},\frac{\partial r}{\partial\eta}\right\|.$$

Since

$$\frac{\partial r^k}{\partial\xi^1}=\frac{\partial\mathring{r}^k}{\partial\xi^1}+\frac{\partial\mathring{r}^k}{\partial s}\frac{\partial s}{\partial\xi^1}, \quad \frac{\partial r^k}{\partial\xi^2}=\frac{\partial\mathring{r}^k}{\partial\xi^2}+\frac{\partial\mathring{r}^k}{\partial s}\frac{\partial s}{\partial\xi^2},$$

$$\frac{\partial r^k}{\partial\eta}=\frac{\partial\mathring{r}^k}{\partial s}\frac{\partial s}{\partial\eta}$$

we have

$$\det\left\|\frac{\partial r}{\partial \xi},\frac{\partial r}{\partial \eta}\right\|=\frac{\partial s}{\partial \eta}\det\left\|\frac{\partial \mathring{r}}{\partial \xi},\frac{\partial \mathring{r}}{\partial s}\right\|.$$

Therefore, from the incompressibility condition, we obtain an equation for the function $s(t,\eta,\xi)$:

$$\frac{\partial s}{\partial \eta}=\sqrt{\mathring{g}}\bigg/\det\left\|\frac{\partial \mathring{r}}{\partial \xi},\frac{\partial \mathring{r}}{\partial s}\right\|. \tag{6.3}$$

This equation determines the function $s(t,\eta,\xi)$, and, thus, the motion of fluid particles on the vortex lines.

VII. STATISTICAL MECHANICS OF POINT VORTICES

Proceeding to the statistical mechanics of ideal fluid flows we start from the 2D case. The phase space of fluid motion is the space of mappings $x=r(\xi)$, $\xi\in V$, $x\in V$. Conservation of energy extracts a surface in phase space:

$$H(r)=\frac{1}{2}\int_V\int_V G(r(\xi),r(\xi'))\mathring{\omega}(\xi)\mathring{\omega}(\xi')d^2\xi d^2\xi'=E$$
$$=\text{const}. \tag{7.1}$$

Dynamical equations of ideal fluid (3.9) possess also a continuum set of integrals owing to the incompressibility condition: for each ξ,

$$\det\left\|\frac{\partial r}{\partial \xi}\right\|=1. \tag{7.2}$$

For point vortex approximation the energy integral (7.1) should apparently be kept, while the incompressibility conditions (7.2) do not seem to be putting constraints on the vortex positions.

Ergodicity of point vortex motion yields [9] statistical independence of positions of any two-point vortices. In the continuum limit any two particles (carrying nonzero vorticity) move independently: for the two-point probability density function one can write

$$f(r,\xi;r',\xi')=f(r,\xi)f(r',\xi'). \tag{7.3}$$

Of course, property (7.3) is an idealization that can be valid only in the limit on an infinite time of observation. For very close particles and/or finite time of observation an incorporation of the correlations might be necessary. However, for such rough characteristics of turbulent motion as the averaged velocity profile an approximation (7.3) may be acceptable; this is supported by the results of Ref. [10].

The probability density function of positions of a particle ξ carrying vorticity $\mathring{\omega}(\xi)$ has been found to be [9]

$$f(r,\xi)=e^{-\beta\mathring{\omega}(\xi)\overline{\psi}(r)}\bigg/\int_V e^{-\beta\mathring{\omega}(\xi)\overline{\psi}(r')}d^2r', \tag{7.4}$$

where $\overline{\psi}(r)$ is the stream function of averaged flow, and β plays the role of inverse temperature. Averaging of the kinematical relation

$$-\Delta\overline{\psi}(t,x)=\omega(t,x)=\int\delta(x-r(t,\xi))\mathring{\omega}(\xi)d^2\xi \tag{7.5}$$

closes the equation for $\overline{\psi}$. One obtains [9]

$$-\Delta\overline{\psi}(r)=\int_V\mathring{\omega}(\xi)\frac{e^{-\beta\mathring{\omega}(\xi)\overline{\psi}(r)}}{\int_V e^{-\beta\mathring{\omega}(\xi)\overline{\psi}(r')}d^2r'}d^2\xi. \tag{7.6}$$

Derivation of the basic relation (7.4) becomes elementary if one accepts the statistical independence of vortex positions (7.3) and the equivalence of microcanonical and canonical distribution. The latter means that the microcanonical distribution of positions $r_1,\dots r_N$ of the vortices $\xi_1,\dots\xi_N$ can be substituted by the Gibbs distribution,

$$f(r_1,\dots,r_N;\xi_1,\dots,\xi_N)=\frac{1}{Z}e^{-\beta H(r_1,\dots,r_N;\xi_1,\dots,\xi_N)},$$

$$Z=\int e^{-\beta H}d^{2N}r. \tag{7.7}$$

Of course, the equivalence takes place only in the limit $N\to\infty$.

To derive Eq. (7.4) from Eqs. (7.1), (7.3), and (7.7) we note that canonical distribution maximizes the entropy considered as a functional of probability density functions,

$$S=-\int f(r_1,\dots,r_N;\xi_1,\dots,\xi_N)$$
$$\times\ln f(r_1,\dots,r_N;\xi_1,\dots,\xi_N)d^{2N}r. \tag{7.8}$$

The set of admissible probability density functions is extracted by the constraints

$$\int f(r_1,\dots,r_N;\xi_1,\dots,\xi_N)H(r_1,\dots,r_N;\xi_1,\dots,\xi_N)d^{2N}r=\overline{E}$$

$$\int f(r_1,\dots,r_N;\xi_1,\dots,\xi_N)d^{2N}r=1. \tag{7.9}$$

Now we modify the variational principle (7.8),(7.9) admitting the additional constraint, the statistical independence of vortex positions:

$$f(r_1,\dots,r_N;\xi_1,\dots,\xi_N)=f(r_1,\xi_1)f(r_2,\xi_2)\cdots f(r_N,\xi_N). \tag{7.10}$$

Then the entropy functional S is transformed to a functional of one-point probability density functions of the form

$$S=-\sum_p\int_V f(r,\xi_p)\ln f(r,\xi_p)d^2r. \tag{7.11}$$

Suppose that region V is covered by a squared grid with cell size ε and where ξ_p are the nodes of the grid; the number of vortices is equal to $|V|/\varepsilon^2+O(\sqrt{|V|}/\varepsilon)$. Multiplying Eq. (7.11) by ε^2 and assuming that $f(r,\xi_p)$ are the values of a smooth function $f(r,\xi)$ at the points ξ_p, we obtain in the left-hand side of Eq. (7.11) an integral sum that converges for $N\to\infty$ to

$$\frac{1}{N}S=-\frac{1}{|V|}\int_V f(r,\xi)\ln f(r,\xi)d^2r d^2\xi. \tag{7.12}$$

The constraints (7.9) take the form

$$\frac{1}{2}\int_V\int_V G(r,r')f(r,\xi)f(r',\xi')\mathring{\omega}(\xi)\mathring{\omega}(\xi') \times d^2r d^2r' d^2\xi d^2\xi d^2\xi' = \overline{E}, \tag{7.13}$$

$$\int_V f(r,\xi)d^2r = 1. \tag{7.14}$$

Maximization of entropy (7.12) with respect to functions $f(r,\xi)$ constrained by Eqs. (7.13) and (7.14) yields the formula (7.4), where β is the Lagrange multiplier for the constraint (7.13).

This means of derivation (7.4) is quite straightforward. The price paid is a number of unjustified hypotheses: statistical independence, equivalence of microcanonical and macrocanonical distributions, and a possibility to change the entropy functional (7.8) by the functional (7.12). The feasibility of such an approach is confirmed by the derivation of Eq. (7.4) from the ergodic hypothesis. In what follows we accept this simple way of constructing the probability measure, leaving the justification for further study.

VIII. PROBABILITY MEASURE FOR 3D FLOWS

Conceptually, the derivation of probability measure for 3D flows is the same as for 2D ones. A technical complication is that the points representing point vortices should be replaced by curves, vortex lines. Denote the ''probability density function'' of positions γ of the vortex line ξ by $f(\gamma,\xi)$. The joint probability density function of N vortex lines is $f(\gamma_1,\xi_1;\cdots\gamma_N,\xi_N)$. We assume the statistical independence of positions of vortex lines:

$$f(\gamma_1,\xi_1;\gamma_2,\xi_2;\cdots\gamma_N,\xi_N) = f(\gamma_1,\xi_1)f(\gamma_2,\xi_2)\cdots f(\gamma_N,\xi_N). \tag{8.1}$$

In the same way as in Sec. VI, this yields the expression for entropy,

$$S = \text{const}\times\int\mathcal{D}\gamma\int f(\gamma,\xi)\ln f(\gamma,\xi)d^2\xi. \tag{8.2}$$

The integration in the space of vortex lines is denoted by $\int\mathcal{D}\gamma$. This integration should be understood as a limit of some finite dimensional truncation of the vortex line dynamics.

In accordance with Eq. (5.4), averaged kinetic energy of the flow is given by

$$E = \frac{1}{2}\int G(\gamma,\gamma')f(\gamma,\xi)\mathring{\omega}(\xi)f(\gamma',\xi')\mathring{\omega}(\xi') \times d^2\xi d^2\xi'\mathcal{D}\gamma\mathcal{D}\gamma'. \tag{8.3}$$

The probability density function is normalized by the condition

$$\int f(\gamma,\xi)\mathcal{D}\gamma = 1. \tag{8.4}$$

Maximizing the functional (8.2) with respect to $f(\gamma,\xi)$ subject to the constraints (8.3) and (8.4) we obtain

$$f(\gamma,\xi) = \text{const}\times\exp\left[-\beta\mathring{\omega}(\xi) \times\int G(\gamma,\gamma')f(\gamma',\xi')\mathring{\omega}(\xi')d^2\xi'\mathcal{D}\gamma'\right]. \tag{8.5}$$

Here β is Lagrange's multiplier for the constraint (8.3). In accordance with Eq. (5.5) the expression in the exponent can be written as

$$\int G(\gamma,\gamma')f(\gamma',\xi')\mathring{\omega}(\xi')d^2\xi' d\gamma' = \mathcal{M}\left[\int G_{ij}(x(\eta),r^k(\eta',\xi'))\frac{\partial x^i(\eta)}{\partial\eta}\frac{\partial r^j(\eta',\xi')}{\partial\eta'} \times\mathring{\omega}(\xi')d\eta d\eta' d^2\xi'\right],$$

where $x = x(\eta)$ is the parametric equation of the curve γ and $\mathcal{M}$ stands for mathematical expectation.

Using Eq. (4.14) the last expression can be put in the form

$$\mathcal{M}_{r'(\xi')}\left[\int G_{ij}(x(\eta),r'(\boldsymbol{\xi}'))\frac{\partial x^i}{\partial\eta}\frac{\partial r^j(\boldsymbol{\xi}')}{\partial\eta'}\mathring{\omega}^a(\boldsymbol{\xi}')d^3\boldsymbol{\xi}' d\eta\right] = \int_\gamma\overline{\psi_i}dx^i,$$

where $\overline{\psi_i}$ is the averaged stream function vector.

Finally,

$$f(\gamma,\xi) = \text{const}\times e^{-\beta\mathring{\omega}(\xi)\int_\gamma\overline{\psi_i}dx^i}. \tag{8.6}$$

Note that additional assumptions specifying integration in the space of the vortex lines should be made in order to fulfill Eq. (8.6) with a mathematical sense. However, a certain observation can be made right away: comparing probability measure (8.6) with the well-studied Wiener measure, which has ''probability density function'' [19,20]

$$f(\gamma) = \text{const}\times e^{-1/2\int_\gamma(dx/d\eta)^2 d\eta}, \tag{8.7}$$

we see that a typical vortex line is less smooth then a typical Wiener curve.

The assumption on statistical independence of the motion of vortex lines puts some severe constraints on the topology of the initial vorticity field. In particular, knotting of the vortex lines is not allowed. The influence of topological invariants of the vorticity field on the probability measure is an interesting open problem.

The topology of the vortex lines seems to be trivial for effectively two-dimensional flows, like Couette's and Poiseuille's flows, and for pipe flows. Therefore, the assumptions made in the derivation of Eq. (8.6) may be meaningful. We consider how to deal with the measure (8.6) for these two cases. Before proceeding to these topics a specification of the relations between the velocity and vorticity for cylindrical regions is needed.

IX. SOME KINEMATICAL RELATIONS FOR CYLINDRICAL REGIONS

General relations of Sec. IV between vorticity, velocity and stream functions can be considerably simplified for cylindrical regions.

Let region V be a cylinder: $V=\{x^1,x^2,x^3:(x^1,x^2)\in\Omega,0\leqslant x^3\leqslant l\}$. The third coordinate plays a distinct role, and we usually drop the index 3, in particular, $x^3\equiv x,\psi^3\equiv\psi,v^3\equiv v,\omega^3\equiv\omega$. Greek indices run values 1,2 and correspond to projections on coordinates x^1,x^2. The set of coordinates x,x^α is denoted by $\mathbf{x}$; similarly, $\mathbf{v}\equiv(v^\alpha,v),\boldsymbol{\omega}\equiv(\omega^\alpha,\omega)$.

It is assumed that at the cylinder surface $\partial\Omega\times[0,l]$ the no-detachment-penetration condition holds:

$$v^\alpha n_\alpha=0 \quad \text{at} \quad \partial\Omega\times[0,l]. \tag{9.1}$$

At the cross sections $x=[0,l]$ periodicity of velocity is posed.

Vorticity is a divergence-free vector field:

$$\frac{\partial\omega}{\partial x}+\frac{\partial\omega^\alpha}{\partial x^\alpha}=0. \tag{9.2}$$

Thus ω component of vorticity is determined by ω^α components and the distribution of ω over the cross section $x=0$:

$$\omega(x,x^\alpha)=\omega_0(x^\alpha)+\frac{\partial}{\partial x^\alpha}\int_0^x\omega^\alpha dx. \tag{9.3}$$

Velocity and vorticity are linked by the relations

$$e^{\alpha\beta}\partial_\alpha v_\beta=\omega, \quad e^{\alpha\beta}(\partial_\beta v-\partial_x v_\beta)=\omega^\alpha. \tag{9.4}$$

It follows from Eqs. (9.4) and (9.1) that the v component of velocity is determined by ω^α components of vorticity:

$$-\Delta_3 v=e^{\alpha\beta}\partial_\alpha\omega_\beta, \quad \frac{\partial v}{\partial n}\equiv n^\beta\partial_\beta v=\omega^\alpha e_{\alpha\beta}n^\beta,$$

$$v(0,x^\alpha)=v(l,x^\alpha), \quad \frac{\partial v}{\partial x}(0,x^\alpha)=\frac{\partial v}{\partial x}(0,x^\alpha). \tag{9.5}$$

Here Δ_3 is 3D Laplace operator.

To write down the solution of the boundary value problem (9.5) in terms of the Green's function it is convenient to put Eq. (9.5) in a weak form: for any smooth periodic in x function ϕ,

$$\int_{\partial\Omega\times[0,l]}v\,\frac{\partial\varphi}{\partial n}\,d^2x-\int_V v\Delta_3\varphi d^3x=\int_V e^{\alpha\beta}\omega_\alpha\,\frac{\partial\varphi}{\partial x^\beta}\,d^3x. \tag{9.6}$$

Equation (9.6) suggests a feasibility to consider the Green's function $H(x,x')$ of the boundary-value problem,

$$\Delta_3 H(x,x')=-\delta(x-x')+\frac{1}{|V|} \quad \text{in} \quad V,$$

$$\frac{\partial H}{\partial x^\alpha}n^\alpha=0 \quad \text{at} \quad \partial\Omega\times[0,l],$$

$$H(0,x^\alpha)=H(l,x^\alpha), \quad \frac{\partial H}{\partial x}(0,x^\alpha)=\frac{\partial H}{\partial x}(l,x^\alpha). \tag{9.7}$$

Putting $\phi=H$ in Eq. (9.6) one obtains

$$v(x,x^\alpha)-\frac{1}{|V|}\int_V v\,d^3x=\int_V e^{\alpha\beta}\omega_\alpha(\mathbf{x}')\,\frac{\partial H(\mathbf{x}',\mathbf{x})}{\partial x'^\beta}\,d^3\mathbf{x}'. \tag{9.8}$$

Other components of velocity can be found from the second equation (9.4),

$$v_\beta(x,x^\alpha)=v_\beta(0,x^\alpha)+\partial_\beta\int_0^x v\,dx-\int_0^x e_{\alpha\beta}\omega^\alpha dx. \tag{9.9}$$

Here v is assumed to be expressed in term of ω^α by means of Eq. (9.8). The 2D vector $v_\beta(0,x^\alpha)$ satisfies equations

$$e^{\alpha\beta}\partial_\alpha v_\beta(0,x^\alpha)=\omega_0(x^\alpha),$$

$$\partial_\alpha v^\alpha(0,x^\beta)=-\frac{\partial v}{\partial x}(0,x^\alpha)$$

$$=-\int_V e^{\alpha\beta}\omega_\alpha(\mathbf{x}')\left.\frac{\partial^2 H(\mathbf{x}',x,x^\beta)}{\partial x'^\beta\partial x}\right|_{x=0}d^3x',$$

$$v^\alpha(0,x^\beta)n_\alpha=0 \tag{9.10}$$

The solution of Eqs. (9.10) can be written in terms of Green's functions of Dirichlet and Neuman problems for cross section Ω.

The ω component of vorticity is a periodic function of x as follows from the first equation (9.4). Thus, the admissible values of ω^α components obey the constraint

$$\frac{\partial}{\partial x^\alpha}\int_0^l\omega^\alpha dx=0, \tag{9.11}$$

which follows from Eq. (9.2).

Note that periodicity condition $v_\beta(0,x^\alpha)=v_\beta(l,x^\alpha)$ is satisfied. Indeed,

$$\partial_\beta\int_0^l v\,dx=\int_0^l dx\int_V e^{\alpha\delta}\omega_\alpha(\mathbf{x}')\,\frac{\partial^2 H(\mathbf{x}',x,x^\beta)}{\partial x'^\delta\partial x^\beta}$$

$$=\int_V e^{\alpha\delta}\omega_\alpha(\mathbf{x}')\,\frac{\partial^2 h(x'^\delta,x^\beta)}{\partial x'^\delta\partial x^\beta}\,d^2x'dx', \tag{9.12}$$

where

$$h(x^\alpha,x'^\alpha)=\int_0^l H(x^\alpha,x;x'^\alpha,x')dx'$$

is the solution of boundary-value problem (Δ_2—2D Laplace operator),

$$\Delta_2 h = -\delta(x^\alpha - x'^\alpha) + \frac{1}{|\Omega|}, \quad \frac{\partial h}{\partial n} = 0 \quad \text{at } \partial\Omega. \tag{9.13}$$

and does not depend on x.

Since, due to Eq. (9.11),

$$\int_0^l \omega^\alpha(\mathbf{x}')dx' = e_{\alpha\beta}\partial_\beta \chi,$$

where χ is some function of x^α,

$$\begin{aligned}
&\int_V e^{\alpha\sigma}\omega_\alpha(\mathbf{x}')\,\frac{\partial^2 h(x'^\sigma,x^\beta)}{\partial x'^\sigma \partial x^\beta}\,d^2x'dx' \\
&= \int_0 \frac{\partial \chi}{\partial x'^\sigma}\,\frac{\partial^2 h(x^{1\sigma},x^\beta)}{\partial x^{1\sigma}\partial x^\beta}\,d^2x' \\
&= -\frac{\partial}{\partial x^\beta}\int \chi(x'^\sigma)\Delta h(x'^\sigma,x^\beta)d^2x' = \frac{\partial \chi(x^\beta)}{\partial x^\beta} \\
&= \frac{\partial}{\partial x^\beta}\int_0^l e_{\alpha\beta}\omega_\alpha(x,x^\beta)dx.
\end{aligned} \tag{9.14}$$

Periodicity of v_α follows from Eqs. (9.10), (9.12), and (9.14).

X. PROBABILISTIC MEASURE FOR CYLINDRICAL REGIONS

Consider the motion of ideal incompressible fluid in a cylindrical region. This motion is assumed to be periodic in the axial direction. An ideal fluid has two additional integrals of motion due to translational symmetry along the axis.

These integrals are

$$\int_V \frac{\partial r^3}{\partial t}\,(\boldsymbol{\xi},t)d^3\boldsymbol{\xi} = \text{const},$$

$$\int_V e_{\alpha\beta} r^\alpha(\boldsymbol{\xi},t)\frac{\partial r^\beta}{\partial \xi^a}\,(\boldsymbol{\xi},t)\,\mathring{\omega}^a(\boldsymbol{\xi})d^3\boldsymbol{\xi} = \text{const}. \tag{10.1}$$

Conservation of the second integral (10.1) can be derived from the invariance of the functional (2.1) with respect to shifts in the axis direction. Kinetic energy in Eq. (2.1) is obviously invariant under such transformation. The functional A is also invariant. Indeed,

$$\begin{aligned}
A(r^\alpha,r^3+c) - A(r^\alpha,r^3) &= \tfrac{1}{3}c\int_V e_{\alpha\beta}v^\alpha\omega^\beta d^3x \\
&= \tfrac{1}{3}c\int_V e_{\alpha\beta}v^\alpha e^{\beta\gamma}(\partial_\gamma v_3 - \partial_3 \nu_\gamma)d^3x \\
&= \tfrac{1}{3}c\int_V (v^\alpha\partial_3 v_\alpha - v^\alpha\partial_\alpha v_3)d^3x.
\end{aligned}$$

The first term is zero due to periodicity of velocity. The second term can be transformed to

$$\int v^\alpha\partial_\alpha v_3 d^3x = \int_0^l dx_3\int_{\partial\Omega} v^\alpha n_\alpha ds - \int_V v_3\partial_\alpha v^\alpha d^3x$$

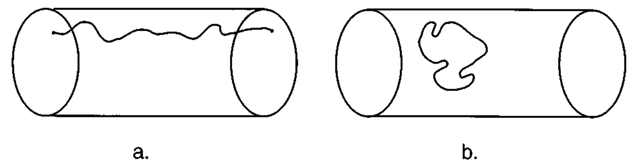

FIG. 1. Typical vortex lines for (a) effectively two dimensional flow (b) pipe flow.

and is also zero because $v^\alpha n_\alpha = 0$ at $\partial\Omega$ and $v_3\partial_\alpha v^\alpha = -v_3\partial_3 v^3$. Therefore,

$$I(r^\alpha,r^3+c) = I(r^\alpha,r^3).$$

Thus,

$$\delta I = 0 \tag{10.2}$$

for any δc. On the other hand,

$$\delta I = -\left[\delta c\,\frac{1}{3}\int e_{\alpha\beta}r^\alpha\,\frac{\partial r^\beta}{\partial \xi^a}\,\mathring{\omega}^a d^3\xi\right]_{t_0}^{t_1} \tag{10.3}$$

for any t_0,t_1. Conservation of the second integral (10.1) follows from Eqs. (10.2) and (10.3).

Additional integrals of motion appear if the cylinder is circular, but we consider here only the general case assuming that even circular cylinders are circular only approximately, and there are small disturbances eliminating rotational symmetry.

Without loss of generality we may use the reference frame in which the total discharge of the flow is equal to zero, and also

$$\int_V r^3(t,\boldsymbol{\xi})d^3\boldsymbol{\xi} = 0. \tag{10.4}$$

Condition (10.4) eliminates the shifts in axial direction.

The second integral (10.1) should be taken into account in maximization of entropy in the form of the constraint

$$\int \mathcal{D}\gamma\int_\gamma e_{\alpha\beta}r^\alpha dr^\beta f(\gamma,\xi)\,\mathring{\omega}(\xi)d^2\xi = \text{const}. \tag{10.5}$$

It is easy to check that the resulting measure is

$$f(\gamma,\xi) = \text{const} \times \exp\left[-\mathring{\omega}(\xi)\left(\beta\int_\gamma \overline{\psi_i}dx^i + \lambda\int_\gamma e_{\alpha\beta}x^\alpha dx^\beta\right)\right], \tag{10.6}$$

where λ is the Lagrange multiplier for the constraint (10.5).

XI. EFFECTIVELY 2D FLOWS

Consider the case when vortex lines are directed at average along the cylinder. This means that each vortex line crosses the planes $x = \text{const}$ [Fig. 1(a)]. We assume for simplicity that the projections of each point of a vortex line on the x axis is unique, and x can be chosen as a parameter along the vortex lines.

Let the average flow be two dimensional, i.e., the averaged components of velocity $\overline{v_\alpha}$ are functions of x^α while $\overline{v_3}=0$. In terms of the stream function vector this corresponds to $\overline{\psi_3}=\psi(x^\alpha)$, $\overline{\psi_\alpha}=0$.

A natural finite-dimensional model of the vortex line would be a set of points at which the vortex line crosses the planes $x=0$, $x=\varepsilon$, $x=2\varepsilon,\ldots$, $x=n\varepsilon=l$. Denoting the projections of these points on the cross section $x=0$ by $\mathbf{r}_0,\ldots,\mathbf{r}_n$ we describe the vortex line by the sequence $\mathbf{r}_1,\ldots,\mathbf{r}_n$ ($\mathbf{r}_0=\mathbf{r}_n$ due to periodicity).

Let first the parameter λ in Eq. (10.6) be zero. Then probability density function of positions $\mathbf{r}_1,\ldots,\mathbf{r}_N$ is

$$f(\mathbf{r}_1,\ldots,\mathbf{r}_n,\xi)=\mathrm{const}\times e^{-\mathring{\omega}(\xi)\beta\varepsilon\Sigma_s\psi(\mathbf{r}_s)}. \tag{11.1}$$

We normalize "coldness" β assuming that $\beta\varepsilon$ tends to some constant for $\varepsilon\to 0$. We keep the notation β for the limit constant. Finally,

$$f(\mathbf{r}_1,\ldots,\mathbf{r}_n,\xi)=\mathrm{const}\times e^{-\mathring{\omega}(\xi)\beta\Sigma_s\psi(\mathbf{r}_s)}. \tag{11.2}$$

Positions of the points $\mathbf{r}_s$ are statistically independent and have the probability function

$$f(r,\xi)=\frac{1}{Z}\,e^{-\mathring{\omega}(\xi)\beta\overline{\psi}(r)},\quad Z=\int e^{-\mathring{\omega}(\xi)\beta\overline{\psi}(r)}d^2r. \tag{11.3}$$

Let us find the equations for the averaged flow. First, we establish that the averaged transversal vorticity $\overline{\omega^\alpha}$ is zero. Indeed, a natural finite-dimensional representation of $\overline{\omega^\alpha}$ at the plane $x=x_s$ is

$$\overline{\omega^\alpha}(x^\beta,x_s)=\overline{\int\delta(x^\beta-r^\beta(x_s,\xi))\,\frac{dr^\alpha(x,\xi)}{dx}\,\mathring{\omega}(\xi)d^2\xi}$$
$$=\overline{\frac{1}{2\varepsilon}\int\delta(x^\beta-r^\beta(x_s,\xi))(r^\alpha_{s+1}-r^\alpha_{s-1})\mathring{\omega}(\xi)d^2\xi}.$$

Since r^α_{s-1}, r^α_s, and r^α_{s+1} are statistically independent, $\overline{r^\alpha_{s-1}}=\overline{r^\alpha_{s+1}}$, $\overline{\omega^\alpha}=0$.

In the same way the averaged axial component of vorticity is obtained:

$$\overline{\omega}(x^\alpha,x_s)=\overline{\int\delta(x^\alpha-r^\alpha(x_s,\xi))\mathring{\omega}(\xi)d^2\xi}$$
$$=\int\delta(x^\alpha-r^\alpha)\,\frac{1}{Z}\,e^{-\mathring{\omega}(\xi)\beta\overline{\psi}(r)}\mathring{\omega}d^2r\,d\xi$$
$$=\int\mathring{\omega}(\xi)\,\frac{e^{-\mathring{\omega}(\xi)\beta\overline{\psi}(r)}}{\int e^{-\mathring{\omega}(\xi)\beta\overline{\psi}(r')}dr'}\,d^2\xi.$$

Since $\overline{\omega}=-\Delta\overline{\psi}$, we arrive at the equation for $\overline{\psi}$ [Eq. (7.6)].

If $\lambda\neq 0$ the situation is more complex because a natural finite-dimensional model for Eq. (10.6) is

$$f(\mathbf{r}_1,\ldots,\mathbf{r}_n,\xi)=\mathrm{const}\times\exp\left[-\mathring{\omega}(\xi)\left(\beta\sum_s\overline{\psi_s}(\mathbf{r}_s)+\lambda\sum_s e_{\alpha\beta}r^\alpha_s r^\beta_{s+1}\right)\right]. \tag{11.4}$$

Positions $\mathbf{r}_s$ are no longer statistically independent.

Parameter λ is determined by the condition

$$\overline{\int e_{\alpha\beta}r^\alpha dr^\beta}=\text{given}. \tag{11.5}$$

Note that the averaged value (11.5) is equal to zero if $\lambda=0$:

$$\overline{\int e_{\alpha\beta}r^\alpha dr^\beta}=\overline{\sum e_{\alpha\beta}r^\alpha_s(r^\beta_{s+1}-r^\beta_{s-1})}=0,$$

since all r_s are statistically independent for $\lambda=0$. If one assumes that there is one-to-one correspondence between the constant (11.5) and parameter λ, the zero value of the constant (11.5) correspond to zero value of λ. Thus, prescribing the zero value of the constant (11.5) we get the measure (11.3).

XII. PIPE FLOW

Consider the flows with one nonzero averaged component of velocity $\overline{v}(x^1,x^2)$, $\overline{v^\alpha}=0$. For such flows $\psi=0$, $\psi^\alpha\neq 0$. Probability measure (10.6) takes the form

$$f(\gamma,\xi)=\mathrm{const}\times\exp\left[-\mathring{\omega}(\xi)\left(\beta\int_{\Omega_\gamma}\overline{v}d^2x+2\lambda\int_{\Omega_\gamma}d^2x\right)\right], \tag{12.1}$$

where Ω_γ is the two-dimensional region bounded by the projection of vortex line γ on the pipe cross section Ω. Denoting the sum $\overline{v}+2\lambda/\beta$ by $\tilde{v}$,

$$\tilde{v}=\overline{v}+\frac{2\lambda}{\beta}, \tag{12.2}$$

we have

$$f(\gamma,\xi)=\mathrm{const}\times e^{-\mathring{\omega}(\xi)\beta\int_{\Omega_\gamma}\tilde{v}d^2x}. \tag{12.3}$$

The probability depends only on projection $x^\alpha=r^\alpha(\eta)$ while all positions of vortex line points along the x axis $x=r(\eta)$ are statistically independent and have equal probability.

Formula (12.3) suggests a natural way to define a finite-dimensional probability measure. Let us cover region Ω by a lattice with the cell centers $\mathbf{r}_s$, $s=1,\ldots,n$ (Fig. 2). Each vortex line projection on Ω is a closed path on the lattice. All projections can be characterized by the set of points $\mathbf{r}_s$ that belong to Ω_γ. Doing that we identify all paths γ and γ' for which the region $\Omega_\gamma-\Omega'_\gamma$ has zero area (Fig. 3).

Each cell r_s either belongs or does not belong to Ω_γ. Therefore, there are 2^n possible projections. They all are statistically independent. The probability that $r_s\in\Omega$ is

$$\mathrm{const}\times e^{-\beta\mathring{\omega}(\xi)\tilde{v}(r_s)\varepsilon^2},$$

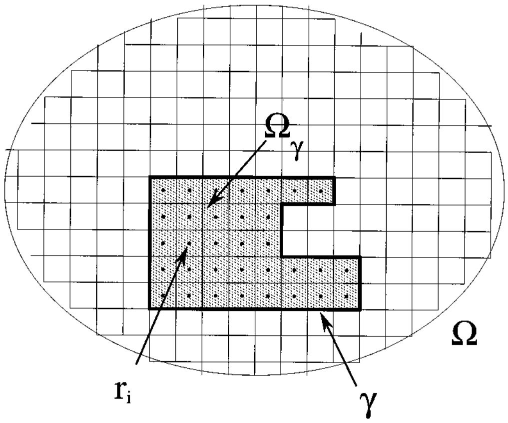

FIG. 2. Vortex line projection.

where $\varepsilon^2 = |\Omega|/N$. "Coldness" is normalized in such a way that $\beta\varepsilon^2 \to \text{const}$ if $\varepsilon \to 0$. Notation β is kept for the limit constant. So,

$$f(\gamma,\xi) = \text{const} \times e^{-\beta \mathring{\omega}(\xi) \Sigma_{s=1}^{n} \tilde{v}(r_s)\eta_s}, \qquad (12.4)$$

where $\eta_s = 1$ if $r_s \in \Omega_\gamma$ and $\eta_s = 0$ otherwise.

The major difference from the case of effectively two-dimensional flow is that, for a typical vortex line, $r(\eta)$ is periodic [Fig. 1(b)]. Therefore the averaged ω component of the vorticity is zero:

$$\overline{\omega(\mathbf{x})} = \overline{\int \delta(\mathbf{x}-\mathbf{r}) \frac{dr}{d\eta} \mathring{\omega} d^2\xi d\eta}$$
$$= \overline{\int \delta(x^\alpha - r^\alpha(\eta))\delta(x - r(\eta)) \frac{dr}{d\eta} \mathring{\omega} d^2\xi d\eta}$$
$$= -\overline{\int \delta(x^\alpha - r^\alpha) \frac{d\theta(x-r)}{d\eta} d\eta \mathring{\omega} d^2\xi} = 0.$$

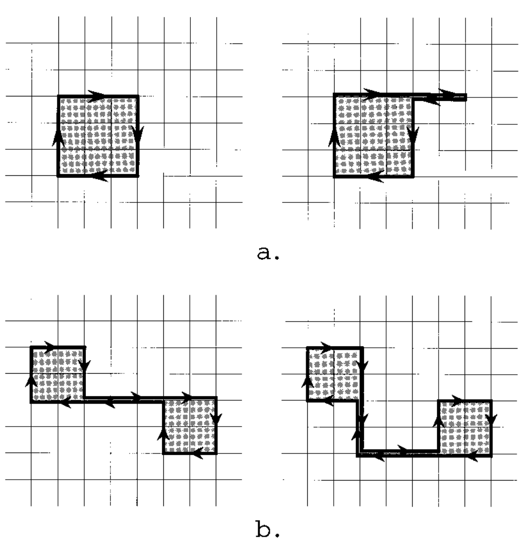

FIG. 3. Examples of identical paths.

Here $d\theta(x)/dx = \delta(x)$, and statistical independence of $r^\alpha(\eta)$ and $r(\eta)$ is used.

To find the equation for $\overline{v}$ consider averaging of Eq. (9.8). We have (bear in mind that we use the frame with zero discharge)

$$\overline{v} = \overline{\int e^{\alpha\beta} \frac{dr^\alpha(\eta,\xi)}{d\eta} \mathring{\omega}(\xi) \frac{\partial H(\mathbf{r},\mathbf{x})}{\partial r^\beta} d^2\xi d\eta}. \qquad (12.5)$$

Since $r(\eta)$ is statistically independent on r^α, averaging with respect to $r(\eta)$ is reduced to integration H over x:

$$\overline{v} = \frac{1}{l} \overline{\int e^{\alpha\beta} \frac{dr^\alpha(\eta,\xi)}{d\eta} \mathring{\omega}(\xi) \frac{\partial h(r,x)}{\partial r^\beta} d^2\xi d\eta}, \qquad (12.6)$$

where h is determined by Eq. (9.13). The right-hand side of Eq. (12.6) can be written as

$$\overline{v} = \frac{1}{l} \overline{\int d^2\xi \mathring{\omega}(\xi) \oint_\gamma e^{\alpha\beta} dr^\alpha \frac{\partial h(r,x)}{\partial r^\beta}}$$
$$= \frac{1}{l} \overline{\int \mathring{\omega}(\xi) d^2\xi \int_{\Omega_\gamma} (-\Delta h) d^2 r}$$
$$= \frac{1}{l} \overline{\int \mathring{\omega}(\xi) d^2\xi \int_{\Omega_\gamma} \left(\delta(x^\alpha - r^\alpha) - \frac{1}{|\Omega|} \right) d^2 r}.$$

The average value of the functional of the form

$$\int_{\Omega_\gamma} G(x^\alpha) d^2 x$$

with respect to measure (12.4) can be easily found:

$$\overline{\int_\Omega \eta(x) G(x) d^2 x} = \overline{\sum_{s=1}^{n} \eta_s G(x_s) \varepsilon^2} = \sum_{s=1}^{n} G(x_s) \varepsilon^2 \overline{\eta_i}$$
$$= \sum_{s=1}^{n} G(x_s) \varepsilon^2 \frac{e^{-\beta\mathring{\omega}(\xi)\tilde{v}(x_s)}}{1 + e^{-\beta\mathring{\omega}(\xi)\tilde{v}(x_s)}}$$
$$= \int_\Omega G(x) \frac{e^{-\beta\mathring{\omega}(\xi)\tilde{v}(x)}}{1 + e^{-\beta\mathring{\omega}(\xi)\tilde{v}(x)}} d^2 x.$$

Finally,

$$\overline{v} = \frac{1}{l} \int \mathring{\omega}(\xi) d^2\xi \Bigg(\frac{e^{-\beta\mathring{\omega}(\xi)\tilde{v}(x)}}{1 + e^{-\beta\mathring{\omega}(\xi)\tilde{v}(x)}}$$
$$- \frac{1}{|\Omega|} \int \frac{e^{-\beta\mathring{\omega}(\xi)\tilde{v}(x)}}{1 + e^{-\beta\mathring{\omega}(\xi)\tilde{v}(x)}} d^2 x \Bigg). \qquad (12.7)$$

In Eq. (12.7) $\tilde{v}$ should be expressed in terms of $\overline{v}$ from Eq. (12.2). The condition of zero discharge determined parameter λ, while the prescribed value of kinetic energy determines parameter β. Equation (12.7) is an integral equation for averaged axial velocity $\overline{v}$. The solution of this equation will be considered elsewhere. One observation, however, can be made right away: it is not seen from the comparison of the equation for pipe flow (12.7) and the equation for effectively two-dimensional flow (7.6) that they are particular cases of some general equations. It might reflect the fact that there are no universal equations of turbulence except that these equations follow from the averaging with respect to an invariant measure that might be universal. If this is the case, the averaged equations should be developed for each class of flow geometry.

APPENDIX A: VARIATION OF FUNCTIONAL (5.6)

Let position vector $r(t,\xi)$ get an infinitesimal variation δr. Variations δr are assumed kinematically consistent at the boundary, i.e.,

$$\delta r^i n_i = 0 \quad \text{at } \partial V. \tag{A1}$$

Variation of the first term in Eq. (5.6) is

$$\begin{aligned}\delta\int_{t_0}^{t_1} A dt &= \int_{t_0}^{t_1}\int_V \frac{1}{3} e_{ijk}[\delta r^i \dot r^j \omega^k - \dot r^i \delta r^j \omega^k - r^i \delta r^j \dot r^k_{,m}\omega^m \\ &\quad + r^i \dot r^j(\delta r^k \omega^m)_{,m}] d^3\xi + \left[\int_V \frac{1}{3} e_{ijk} r^i \delta r^j \omega^k d^3\xi\right]_{t_0}^{t_1} \\ &= \int_{t_0}^{t_1}\int_V e_{ijk}\delta r^i \dot r^j \omega^k d^3\xi \\ &\quad + \left[\int_V \frac{1}{3} e_{ijk} r^i \delta r^j \omega^k d^3\xi\right]_{t_0}^{t_1}. \end{aligned}\tag{A2}$$

For variation of kinetic energy (5.3) we have

$$\begin{aligned}\delta K = \int_V\int_V \Bigg[&\frac{\partial \widetilde{G}_{ij}[r, r(t,\boldsymbol{\xi}')]}{\partial r^k}\bigg|_{r=r(t,\boldsymbol{\xi})} \delta r^k \frac{\partial r^i(t,\boldsymbol{\xi})}{\partial \xi^a} \mathring{\omega}^a(\boldsymbol{\xi}) \\ &\times \frac{\partial r^j(t,\boldsymbol{\xi}')}{\partial \xi'^b}\mathring{\omega}^b(\boldsymbol{\xi}') + \widetilde{G}_{ij}[r(t,\boldsymbol{\xi}), r(t,\boldsymbol{\xi}')] \\ &\times \frac{\partial \delta r^i}{\partial \xi^a}\mathring{\omega}^a(\boldsymbol{\xi}) \frac{\partial r^j(t,\boldsymbol{\xi}')}{\partial \xi'^b}\mathring{\omega}^b(\boldsymbol{\xi}')\Bigg] d^3\xi d^3\xi'.\end{aligned}$$

Changing integration over Lagrangian coordinates to integration over Eulerian coordinates we get

$$\begin{aligned}\delta K = \int_V\int_V \Bigg[&\frac{\partial \widetilde{G}_{ij}(x,x')}{\partial x^k}\delta r^k(t,x)\omega^i(t,x)\omega^j(t,x') \\ &+ \widetilde{G}_{ij}(x,x')\delta r^i_{,k}(t,x)\omega^k(t,x)\omega^j(t,x')\Bigg] d^3x d^3x'.\end{aligned}\tag{A3}$$

In accordance with Eq. (4.18) the first term in Eq. (A3) can be written as

$$\begin{aligned}&\int_V\int_V \frac{\partial \widetilde{G}_{ij}(x,x')}{\partial x^k}\delta r^k(t,x)\omega^i(t,x)\omega^j(t,x')d^3x d^3x' \\ &= \int_V\int_V\int_V \frac{\partial R_{mi}(\widetilde{x},x)}{\partial x^k} R^m_j(\widetilde{x},x')\delta r^k(t,x)\omega^i(t,x) \\ &\quad \times \omega^j(t,x')d^3x d^3x' d^3\widetilde{x} \\ &= \int_V\int_V \frac{\partial R_{mi}(\widetilde{x},x)}{\partial x^k} v^m(t,\widetilde{x})\delta r^k(t,x)\omega^i(t,x)d^3x d^3\widetilde{x}.\end{aligned}\tag{A4}$$

By the same reasonings, the second term in Eq. (A3) can be transformed to

$$\begin{aligned}&\int_V\int_V \widetilde{G}_{ij}(x,x')\delta r^i_{,k}(t,x)\omega^k(t,x)\omega^j(t,x')d^3x d^3x' \\ &= \int_V\int_V\int_V R_{mi}(\widetilde{x},x)R^m_j(\widetilde{x},x')\delta r^i_{,k}(t,x)\omega^k(t,x) \\ &\quad \times \omega^j(t,x')d^3x d^3x' d^3\widetilde{x} \\ &= \int_V\int_V R_{mi}(\widetilde{x},x)v^m(t,\widetilde{x})\delta r^i_{,k}(t,x)\omega^k(t,x)d^3x d^3\widetilde{x} \\ &= -\int\int \frac{\partial R_{mi}(\widetilde{x},x)}{\partial x^k} v^m(t,\widetilde{x})\delta r^i(t,x)\omega^k(t,x)d^3x d^3\widetilde{x}.\end{aligned}\tag{A5}$$

Combining (A3)–(A5) we get

$$\delta K = \int_V (\delta r^k \omega^i - \delta r^i \omega^k)\frac{\delta\theta_i}{\partial x^k} d^3x, \tag{A6}$$

where

$$\theta_i(t,x) = \int_V R_{mi}(\widetilde{x},x)v^m(t,\widetilde{x})d^3x.$$

Denote the difference $\theta_i - \psi_i$ by μ_i. In accordance with Eq. (4.6),

$$\mu_i \equiv \theta_i - \psi_i = \int [R_{mi}(\widetilde{x},x) - R_{im}(x,\widetilde{x})]v^m(t,\widetilde{x})d^3\widetilde{x}.$$

Identity (4.11) yields

$$\int_V \mu_i \varepsilon^i d^3x = 0$$

for any divergence-free vector field ε^i.

Substituting $\theta_i = \psi_i + \mu_i$ into Eq. (A6), we see that the term containing μ_i vanishes, because, in accordance with Eqs. (5.6) and (4.8),

$$\begin{aligned}&\int_V (\delta r^k \omega^i - \delta r^i \omega^k)\frac{\partial \mu_i}{\partial x^k} d^3x \\ &= -\int \mu_i \frac{\partial}{\partial x^k}(\delta r^k \omega^i - \delta r^i \omega^k)d^3x,\end{aligned}$$

and vector $(\delta r^k\omega^i-\delta r^i\omega^k)_{,k}$ is divergence free.

Finally

$$\delta K=\int_V(\delta r^k\omega^i-\delta r^i\omega^k)\psi_{i,k}d^3x=2\int_V\delta r^k\omega^i\psi_{[i,k]}d^3x$$
$$=\int_V\delta r^k\omega^i e_{ikj}v^j d^3x. \tag{A7}$$

Here $\psi_{[i,k]}\equiv 1/2(\psi_{i,k}-\psi_{k,i})$ and we used the relation $\psi_{[i,k]}=1/2e_{ikj}\omega^j$.

Assume that the positions of particles are prescribed at $t=t_0,t_1$. Then $\delta r^i=0$ at $t=t_0,t_1$ and the last term in Eq. (A2) vanishes (in fact, to vanish this term, a weaker constraints may be set). Combining Eqs. (A2) and (A7) we arrive at the expression for variation of functional I [Eq. (5.7)].

APPENDIX B: SYMMETRIES OF THE ACTION FUNCTIONAL AND INTEGRALS OF THE MOTION

In this appendix the groups of symmetries of the action functional are found. They cause the extra integrals of motion of vortex dynamics to exist. We determine the corresponding integrals of motion in the dynamics of the vortex lines. We begin our consideration with the discussion of the well-known fact that the conservation of the velocity circulations stems from the invariance of kinetic energy with respect to the relabeling group of transformations.

1. Relabeling group

Consider the Hamilton variational principle: true motion of an ideal incompressible fluid is a stationary point of the action functional,

$$I(x(t,\xi))=\int_{t_0}^{t_1}\int_V\frac{1}{2}\rho\,\frac{\partial x^i(t,\xi)}{\partial t}\,\frac{\partial x_i(t,\xi)}{\partial t}\,d^3\xi, \tag{B1}$$

on the set of all functions $x(t,\xi)$ such that their initial and final values are prescribed,

$$x(t_0,\xi)=x_0(\xi),\quad x(t_1,\xi)=x_1(\xi), \tag{B2}$$

fluid does not detach from or penetrate through the wall,

$$x(t,\xi)\in V\quad\text{if}\quad\xi\in V, \tag{B3}$$

and the motion is incompressible,

$$\left|\frac{\partial x}{\partial\xi}\right|=1. \tag{B4}$$

In this section, we do not need to mark the vortex lines, thus here ξ denotes the set of all three Lagrangian coordinates. Let us rename the particles: $\xi\to\eta(\xi)$, and, for a given motion,

$$x=x(t,\xi),$$

consider another motion,

$$x=x'(t,\xi)\equiv x(t,\eta(\xi)). \tag{B5}$$

Condition (B3) is obviously satisfied. To satisfy Eq. (B4) we set

$$\left|\frac{\partial\eta}{\partial\xi}\right|=1. \tag{B6}$$

The new motion (B5) does not obey Eq. (B2), but this is not necessary for our purposes.

The action functional has the same values for both motions, $x=x(t,\xi)$ and $x=x'(t,\xi)$. Indeed,

$$I(x'(t,\xi))=\int_{t_0}^{t_1}\int_V\frac{1}{2}\rho\,\frac{\partial x'^i(t,\xi)}{\partial t}\,\frac{\partial x'_i(t,\xi)}{\partial t}\,d^3\xi$$
$$=\int_{t_0}^{t_1}\int_V\frac{1}{2}\rho\,\frac{\partial x^i(t,\eta(\xi))}{\partial t}\,\frac{\partial x_i(t,\eta(\xi))}{\partial t}\,d^3\xi$$
$$=\int_{t_0}^{t_1}\int_V\frac{1}{2}\rho\,\frac{\partial x^i(t,\eta)}{\partial t}\,\frac{\partial x_i(t,\eta)}{\partial t}\left|\frac{\partial\xi}{\partial\eta}\right|d^3\eta. \tag{B7}$$

Taking into account Eq. (B6) and changing the notation for the integration variables from η to ξ we see that the integral (B7) coincide with the integral (B1). Therefore,

$$\delta I=I(x'(t,\xi))-I(x(t,\xi))\equiv 0. \tag{B8}$$

Let relabeling be infinitesimal, i.e., $\eta=\xi+\delta\xi$. Then

$$\delta x^i=x'^i(t,\xi)-x^i(t,\xi)=\frac{\partial x^i}{\partial\xi^a}\,\delta\xi^a. \tag{B9}$$

The variation of functional (B1) is

$$\delta I=\int_{t_0}^{t_1}\int_V\rho v_i\,\frac{\partial\delta x^i}{\partial t}\,d^3\xi=\left[\int_V\rho v_i\delta x^i d^3\xi\right]_{t_0}^{t_1}$$
$$-\int_{t_0}^{t_1}\int_V\left[\frac{\partial}{\partial t}\,\rho v_i(t,\xi)\right]\delta x^i d^3\xi. \tag{B10}$$

Assume that the motion $x(t,\xi)$ obeys the Euler equations. Then the last integral in Eq. (B10) vanishes because

$$\int_{t_0}^{t_1}\int_V\left[\frac{\partial}{\partial t}\,\rho v_i(t,\xi)\right]\delta x^i d^3\xi=\int_{t_0}^{t_1}\int_V\left(-\frac{\partial p}{\partial x^i}\,\delta x^i\right)d^3x$$
$$=-\int_{t_0}^{t_1}\int_{\partial V}p\,\delta x^i n_i d^2x=0. \tag{B11}$$

Here we used the Euler equations (2.1) and integrated by parts taking into account that

$$\frac{\partial\delta x^i}{\partial x^i}=0,$$

due to Eq. (B4), and that

$$\delta x^i n_i=0,$$

due to Eq. (B3) and the way in which the motion $x'(t,\xi)$ was defined. Note that Eqs. (B11) and (B8) hold for any t_0, t_1. Thus, in accordance with (B8), (B10), and (B11), for any t_0, t_1,

$$\int_V \rho v_i \delta x^i d^3\xi \bigg|_{t=t_0} = \int_V \rho v_i \delta x^i d^3\xi \bigg|_{t=t_1}$$

or

$$\int_V \rho v_i \frac{\partial x^i}{\partial \xi^a} \delta\xi^a d^3\xi \bigg|_{t=t_0} = \int_V \rho v_i \frac{\partial x^i}{\partial \xi^a} \delta\xi^a d^3\xi \bigg|_{t=t_1}. \tag{B12}$$

Functions $\delta\xi^a$ are not arbitrary. Due to Eq. (B6) they obey the equation

$$\frac{\partial \delta\xi^a}{\partial \xi^a} = 0.$$

This equation is satisfied if $\delta\xi^a$ is a vector field concentrated at any closed fluid line γ and having a constant projection on this line (the situation is similar to that for vorticity in Sec. V). Denoting a parameter along the line by η one obtains from Eq. (B12)

$$\int_\gamma \rho v_i \frac{\partial x^i}{\partial \eta} d\eta \bigg|_{t=t_0} = \int_\gamma \rho v_i \frac{\partial x^i}{\partial \eta} d\eta \bigg|_{t=t_1} = \int_\gamma \rho v_i dx^i,$$

i.e., the conservation of the velocity circulation along any closed fluid contour.

2. Isovorticity group in 2D

The variational principles for the functionals (3.10) and (5.6) differ from the Hamilton variational principle by elimination of many symmetries and, consequently, many integrals of motion (the velocity circulations). Nevertheless, certain symmetries (and integrals of motion) still remain. For example, 2D dynamical equations (3.9) yield the conservation of the particle volume (at each material point ξ, $|\partial x/\partial \xi| = \text{const}$). We arrive at the following questions: What is the underlying group of symmetry for these integrals in 2D? What are the symmetry group and the corresponding integrals in 3D? Here we show that, for the two-dimensional case, this is the group relabeling the particles with the same vorticity. We call it the isovorticity group. More precisely, consider the action functional for two-dimensional flows:

$$I(r) = \int_{t_0}^{t_1} dt \left[\int_V y(t,\xi) \frac{\partial x(t,\xi)}{\partial t} \mathring{\omega}(\xi) d^2\xi - K \right],$$

$$K = \frac{1}{2} \int_V \int_V G(r(t,\xi), r(t,\xi')) \mathring{\omega}(\xi) \mathring{\omega}(\xi') d^2\xi d^2\xi'. \tag{B13}$$

Let us show that the action functional has the same value for two motions $x = r(t,\xi)$ and $x = r'(t,\xi)$ if

$$r'(t,\xi) = r(t, \eta(\xi)),$$

and the relabeling $\xi \rightarrow \eta(\xi)$ conserves the vorticity:

$$\mathring{\omega}(\xi) = \mathring{\omega}(\eta(\xi)) \left| \frac{\partial \eta}{\partial \xi} \right|. \tag{B14}$$

Equation (B14) can be written also in the form

$$\mathring{\omega}(\xi) d^2\xi = \mathring{\omega}(\eta) d^2\eta,$$

emphasizing the vorticity conservation as a measure. Incompressibility of the motion follows from this symmetry group. Indeed, the kinetic energy is an invariant under such transformation:

$$\begin{aligned} K(r'(t,\xi)) &= \frac{1}{2} \int_V \int_V G(r'(t,\xi), r'(t,\tilde{\xi})) \mathring{\omega}(\xi) \mathring{\omega}(\tilde{\xi}) d^2\xi d^2\tilde{\xi} \\ &= \frac{1}{2} \int_V \int_V G(r(t,\eta(\xi)), r(t,\eta(\tilde{\xi}))) \mathring{\omega}(\xi) \mathring{\omega}(\tilde{\xi}) \\ &\quad \times d^2\xi d^2\tilde{\xi} \\ &= \frac{1}{2} \int_V \int_V G(r(t,\eta), r(t,\tilde{\eta})) \mathring{\omega}(\xi(\eta)) \mathring{\omega}(\xi(\tilde{\eta})) \\ &\quad \times \left| \frac{\partial \xi}{\partial \eta} \right| \left| \frac{\partial \xi}{\partial \tilde{\eta}} \right| d^2\eta d^2\tilde{\eta} \\ &= \frac{1}{2} \int_V \int_V G(r(t,\eta), r(t,\tilde{\eta})) \mathring{\omega}(\omega) \mathring{\omega}(\tilde{\eta}) d^2\eta d^2\tilde{\eta} \\ &= K(r(t,\xi)). \end{aligned}$$

Here we used the fact that Eq. (B14) can be written also as

$$\mathring{\omega}(\xi(\eta)) \left| \frac{\partial \xi}{\partial \eta} \right| = \mathring{\omega}(\eta).$$

The first integral in Eq. (B13) is also the invariant:

$$\begin{aligned} &\int_V y'(t,\xi) \frac{\partial x'(t,\xi)}{\partial t} \mathring{\omega}(\xi) d^2\xi \\ &\quad = \int_V y(t,\eta(\xi)) \frac{\partial x(t,\eta(\xi))}{\partial t} \mathring{\omega}(\xi) d^2\xi \\ &\quad = \int_V y(t,\eta) \frac{\partial x(t,\eta)}{\partial t} \mathring{\omega}(\xi(\eta)) \left| \frac{\partial \xi}{\partial \eta} \right| d^2\eta \\ &\quad = \int_V y(t,\eta) \frac{\partial x(t,\eta)}{\partial t} \mathring{\omega}(\eta) d^2\eta. \end{aligned}$$

In the same way as for the Hamilton variational principle we obtain that

$$\begin{aligned} &\int_V y(t,\xi) \delta x(t,\xi) \mathring{\omega}(\xi) d^2\xi \\ &\quad = \int_V y(t,\xi) \frac{\partial x(t,\xi)}{\partial \xi^\mu} \delta\xi^\mu \mathring{\omega}(\xi) d^2\xi = \text{const}. \end{aligned} \tag{B15}$$

The functions $\delta\xi^\mu$ in Eq. (B15) obey the constraint that follows from Eq. (B14),

$$\mathring{\omega}(\xi)=\mathring{\omega}(\xi+\delta\xi)\left|\frac{\partial(\xi+\delta\xi)}{\partial\xi}\right|$$
$$=\left(\mathring{\omega}(\xi)+\frac{\partial\mathring{\omega}}{\partial\xi^\mu}\,\delta\xi^\mu\right)\left(1+\frac{\partial\delta\xi^\mu}{\partial\xi^\mu}\right).$$

Keeping only the leading terms we obtain

$$\frac{\partial\mathring{\omega}\,\delta\xi^\mu}{\partial\xi^\mu}=0.$$

Thus,

$$\mathring{\omega}\,\delta\xi^\mu=e^{\mu\nu}\,\frac{\partial\chi(\xi)}{\partial\xi^\nu},\tag{B16}$$

where χ is an arbitrary function. The function χ should be equal to zero at the boundary of simply connected regions if $\mathring{\omega}\neq 0$ at the boundary (that is assumed for simplicity).

Plugging Eq. (B16) into Eq. (B15) and integrating by parts we obtain

$$\int_V e^{\mu\nu}\,\frac{\partial y(t,\xi)}{\partial\xi^\mu}\,\frac{\partial x(t,\xi)}{\partial\xi^\nu}\,\chi(\xi)\,d^2\xi=\text{const}.$$

Since $\chi(\xi)$ is an arbitrary function, $|\partial x/\partial\xi|=\text{const}$ at each particle, as was claimed.

3. Isovorticity group in 3D

The symmetry group of the functional (5.6) is a relabeling group that conserves vorticity in the following sense:

$$\mathring{\omega}^a(\xi)\sqrt{\mathring{g}(\xi)}\,\frac{\partial\eta^b(\xi)}{\partial\xi^a}=\mathring{\omega}^b(\eta(\xi))\sqrt{\mathring{g}(\eta(\xi))}\left|\frac{\partial\eta}{\partial\xi}\right|.$$

This can be checked by inspection. For infinitesimal transformation, the admissible variations $\delta\xi^a=\eta^a-\xi^a$ obey the equation

$$\frac{\partial}{\partial\xi^a}\left[\sqrt{\mathring{g}}(\mathring{\omega}^a\,\delta\xi^b-\mathring{\omega}^b\,\delta\xi^a)\right]=0.$$

The general solution of this equation is

$$\mathring{\omega}^a\,\delta\xi^b-\mathring{\omega}^b\,\delta\xi^a=\frac{1}{\sqrt{\mathring{g}}}\,e^{abc}\,\frac{\partial\chi(\xi)}{\partial\xi^c}.\tag{B17}$$

Equation (B17) can be resolved with respect to $\partial\chi/\partial\xi^c$:

$$\frac{\partial\chi}{\partial\xi^c}=\sqrt{\mathring{g}}\,e_{abc}\,\mathring{\omega}^a\,\delta\xi^b.\tag{B18}$$

If $\delta\xi^a$ is proportional to $\mathring{\omega}^a$, i.e., one relabels the particles on the same vortex lines, then $\partial\chi/\partial\xi^c\equiv 0$. We consider the symmetries with respect to the relabeling of the neighboring vortex lines, i.e., $\delta\xi^a\neq\lambda\,\mathring{\omega}^a$ at all points.

Both vectors $\mathring{\omega}^a$ and $\delta\xi^a$ are tangent to the boundary. Projecting Eq. (B18) on the tangent directions to the boundary we obtain that χ is constant on the boundary, and, without loss of generality, can be set equal to zero. It also follows from Eq. (B18) that vectors $\partial\chi/\partial\xi^c$ and $\mathring{\omega}^c$ are orthogonal:

$$\mathring{\omega}^c\,\frac{\partial\chi}{\partial\xi^c}=0.\tag{B19}$$

In the same way as in the previous two cases, from the invariance of the action functional we obtain:

$$\int e_{ijk}x^i(t,\xi)\,\delta x^j(t,\xi)\,\frac{\partial x^k(t,\xi)}{\partial\xi^a}\,\mathring{\omega}^a(\xi)\sqrt{\mathring{g}}\,d^3\xi=\text{const}.$$

Here

$$\delta x^j=\frac{\partial x^j}{\partial\xi^b}\,\delta\xi^b.$$

Hence,

$$\int e_{ijk}x^i\,\frac{\partial x^j}{\partial\xi^a}\,\frac{\partial x^k}{\partial\xi^b}\,\delta\xi^b\,\mathring{\omega}^a\sqrt{\mathring{g}}\,d^3\xi$$
$$=\int e_{ijk}x^i\,\frac{\partial x^j}{\partial\xi^a}\,\frac{\partial x^k}{\partial\xi^b}\frac{1}{2}(\mathring{\omega}^a\,\delta\xi^b-\mathring{\omega}^b\,\delta\xi^a)\sqrt{\mathring{g}}\,d^3\xi$$
$$=\int e_{ijk}x^i\,\frac{\partial x^j}{\partial\xi^a}\,\frac{\partial x^k}{\partial\xi^b}\,e^{abc}\,\frac{\partial\chi}{\partial\xi^c}\,d^3\xi.$$

Integrating by parts, we obtain the following expression for this integral:

$$-\int e_{ijk}\,\frac{\partial x^i}{\partial\xi^c}\,\frac{\partial x^j}{\partial\xi^a}\,\frac{\partial x^k}{\partial\xi^b}\,e^{abc}\chi(\xi)\,d^3\xi=-3!\int\left|\frac{\partial x}{\partial\xi}\right|\chi(\xi)\,d^3\xi.$$

In the vortex line coordinate system χ is constant along the vortex lines. Function χ is arbitrary as a function of the vortex line. Thus, for for each vortex line, the integral

$$\int\left|\frac{\partial x}{\partial\xi}\right|d\eta=\text{const}\tag{B20}$$

remains unchanged in the course of motion.

The existence of this invariant of motion can be derived directly from Eq. (5.10). Indeed, differentiating this equation with respect to x^i and taking into account that, due to Eq. (5.10), $\partial V^i/\partial x^i=0$, we obtain

$$\frac{\partial\dot{r}^i}{\partial x^i}=\frac{\partial(\lambda x^i_a\mathring{\omega}^a)}{\partial x^i}=\frac{\partial(\lambda\,\mathring{\omega}^a)}{\partial x^i}\,\frac{\partial x^i}{\partial\xi^a}+\lambda\,\mathring{\omega}^a\,\frac{\partial x^i_a}{\partial x^i}=\frac{\partial\lambda\,\mathring{\omega}^a}{\partial\xi^a}$$
$$+\lambda\,\mathring{\omega}^a\,\frac{\partial^2x^i}{\partial\xi^a\partial\xi^b}\,\frac{\partial\xi^b}{\partial x^i}.$$

Since

$$\frac{\partial\xi^b}{\partial x^i}=\frac{1}{|\partial x/\partial\xi|}\,\frac{\partial|\partial x/\partial\xi|}{\partial(\partial x^i/\partial\xi^b)},$$

we have

$$\frac{\partial^2x^i}{\partial\xi^a\partial\xi^b}\,\frac{\partial\xi^b}{\partial x^i}=\frac{\partial^2x^i}{\partial\xi^a\partial\xi^b}\,\frac{1}{|\partial x/\partial\xi|}\,\frac{\partial|\partial x/\partial\xi|}{\partial(\partial x^i/\partial\xi^b)}$$

$$=\frac{1}{|\partial x/\partial\xi|}\frac{\partial|\partial x/\partial\xi|}{\partial\xi^a}.$$

So,

$$\frac{\partial\dot{r}^i}{\partial x^i}=\frac{\partial(\lambda\mathring{\omega}^a)}{\partial\xi^a}+\lambda\mathring{\omega}^a\frac{1}{|\partial x/\partial\xi|}\frac{\partial|\partial x/\partial\xi|}{\partial\xi^a}$$
$$=\frac{1}{|\partial x/\partial\xi|}\frac{\partial}{\partial\xi^a}\left(\lambda\mathring{\omega}^a\left|\frac{\partial x}{\partial\xi}\right|\right). \tag{B21}$$

On the other hand,

$$\left.\frac{\partial}{\partial t}\right|_{\xi=\mathrm{const}}\left|\frac{\partial x}{\partial\xi}\right|=\frac{\partial|\partial x/\partial\xi|}{\partial x_a^i}\frac{\partial}{\partial t}\frac{\partial x^i}{\partial\xi^a}=\left|\frac{\partial x}{\partial\xi}\right|\frac{\partial\xi^a}{\partial x^i}\frac{\partial\dot{r}^i}{\partial\xi^a}=\left|\frac{\partial x}{\partial\xi}\right|\frac{\partial\dot{r}^i}{\partial x^i}. \tag{B22}$$

Combining Eqs. (B21) and (B22), we find

$$\left.\frac{\partial}{\partial t}\right|_{\xi=\mathrm{const}}\left|\frac{\partial x}{\partial\xi}\right|=\frac{\partial}{\partial\xi^a}\left(\lambda\mathring{\omega}^a\left|\frac{\partial x}{\partial\xi}\right|\right). \tag{B23}$$

Equation (B23) can be also written as

$$\left.\frac{\partial}{\partial t}\right|_{\xi=\mathrm{const}}\left|\frac{\partial x}{\partial\xi}\right|=\sqrt{\mathring{g}}\,\mathring{\omega}^a\frac{\partial}{\partial\xi^a}\left(\lambda\frac{|\partial x/\partial\xi|}{\sqrt{\mathring{g}}}\right). \tag{B24}$$

Here we used Eq. (5.1). In the vortex line coordinate system Eq. (B24) takes the form

$$\left.\frac{\partial}{\partial t}\right|_{\xi=\mathrm{const}}\left|\frac{\partial x}{\partial\xi}\right|=\mathring{\omega}\frac{\partial}{\partial\eta}\left(\lambda\frac{|\partial x/\partial\xi|}{\sqrt{\mathring{g}}}\right). \tag{B25}$$

The vortex intensity $\mathring{\omega}$ does not depend on η. Integrating Eq. (B25) over a closed vorticity line we obtain the integral of motion,

$$\frac{\partial}{\partial t}\int\left|\frac{\partial x}{\partial\xi}\right|\partial\eta=0.$$

The invariance of the action functional with respect to relabeling the particles on the same vortex line produces a ''degeneracy'' of Eq. (5.10): contracting of (5.10) with the vorticity vector gives an identity.

The integrals (B20) mean a kind of ''two-dimensional incompressibility'' for the vortex line dynamics. These integrals do not constrain the motion of a finite set of vortex lines, and, thus, does not affect the probability measure (1.1).

APPENDIX C: VISCOSITY AND INTEGRALS OF MOTION

Successful prediction of turbulent velocity profiles in Couette and Poiseuille flows by statistical mechanics of point vortices [10] raises two questions: (i) Why can essentially three-dimensional dynamics be described by a theory of point vortices that is two-dimensional? (ii) Why does the theory of ideal fluid work for flows bounded by the walls while it is well known that viscosity contributes essentially in fluid dynamics near the walls? The answer to the first question is given by the consideration in Sec. XI: point vortices can be considered as ''averaged images'' of curvilinear three-dimensional vortex lines, and the laws of statistical mechanics of point vortices stay valid for these ''averaged images.'' In this appendix the second question is discussed.

Consider, for definiteness, the Couette flow of viscous incompressible fluid between two parallel walls. Walls move in opposite directions with velocities u and $-u$. The wall velocity is assumed to be large, so some steady turbulent regime is developed. One may think that in phase space the phase trajectory moves along the attractor of viscous fluid. Experiments show that fluctuations of total energy of the flow are small, of order percent. This means that the attractor lies in a small vicinity of the surface of constant energy. It seems natural to try to approximate the motion on the attractor by the motion of an ideal fluid flow over the energy surface. The motion of ideal fluid is not ergodic on the energy surface: each trajectory belongs to a sheet on energy surface extracted by the values of initial vorticity. Assuming that motion on the sheet is ergodic, one can try to approximate the invariant measure of the attractor by the invariant measure of some sheet. To determine a sheet corresponding to the attractor one has to give a recipe to establish the values of initial vorticity and energy for this sheet. A way to do that has been proposed in Ref. [10]. Let a snapshot of some turbulent vorticity field be made. Consider the dynamics of an ideal fluid with the initial vorticity field obtained from the snapshot. The question is: Will the trajectory of ideal fluid be statistically close to the trajectory of a viscous fluid? Or, in other words: If at some instant viscosity is set equal to zero, will the motion of an ideal fluid and viscous fluid be statistically close? At first glance, the answer is no because ideal fluid motion does not satisfy no-slip boundary conditions. Fortunately enough, however, this does not seem to be a real obstacle: integrals of motion help to maintain no-slip boundary conditions in the average. For example, for Couette's flow total vorticity $\int\omega\,dxdy$ is conserved. The total vorticity can be written in the form

$$\int\omega\,dxdy=-\int_0^l u_+dx+\int_0^l u_-dx, \tag{C1}$$

where the y coordinate is orthogonal to the walls, u_+ and u_- are the values of the x component of velocity at the walls, and the periodicity condition is imposed in the x direction. Therefore, conservation of vorticity makes the flow of ideal fluid to keep the initial difference between wall velocities in average.

For Poiseuille's flow the role similar to total vorticity (C1) is played by the integral of motion

$$\int y\omega \, dxdy. \tag{C2}$$

This might be an explanation of why the statistical mechanics of ideal fluid gives the experimentally observed velocity profiles for Couette's and Poiseuille's flows. If the wall geometry is more complex the contribution of viscosity in the averaged equations may be important.

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